Final Exam

MATH 106-D, Fall 2015

Name: ____

Instructions:

- Please answer as many of the following questions as possible.
- No cell phones or collaboration allowed. If you leave the classroom during the exam you must leave your cell phone with the instructor.
- Approved calculators are allowed.
- Additional scrap paper is available upon request.
- Show all of your work on the page of the problem. Clearly indicate your answer and the reasoning that you used to arrive at the answer. You do not have to simplify algebraic expressions.

This exam has 7 problems. There are a total of 100 points.

Good luck!

Problem	**Possible Points**	**Points Earned**
1	18	
2	12	
3	16	
4	12	
5	12	
6	16	
7	14	
TOTAL	**100**	

1. (18 points) Evaluate the following integrals. Show all of your work and do not use Taylor series.

   (a) $\displaystyle\int \frac{e^{2x}}{3(e^{2x}-3)}\,dx$

   SOLUTION: Use substitution, $u = e^{2x} - 3$, $du = 2e^{2x}\,dx$

$$\begin{aligned}\int \frac{e^{2x}}{3(e^{2x}-3)}\,dx &= \frac{1}{6}\int \frac{du}{u} \\ &= \frac{1}{6}\ln|u| + C \\ &= \frac{1}{6}\ln|e^{2x}-3| + C\end{aligned}$$

   (b) $\displaystyle\int (2x+1)\cos(x)\,dx$

   SOLUTION: Use integration by parts,

$$\begin{array}{ll} u = 2x+1 & dv = \cos x\,dx \\ du = 2\,dx & v = \sin x \end{array}$$

$$\begin{aligned}\int (2x+1)\cos x\,dx &= (2x+1)\sin x - \int 2\sin x\,dx \\ &= (2x+1)\sin x + 2\cos x + C\end{aligned}$$

(c) $\int \frac{dx}{(x^2+1)^{3/2}}$

SOLUTION: Use trig substitution. Let $x = \tan\theta$. Then $dx = \sec^2\theta\, d\theta$.

$$\begin{aligned}
\int \frac{dx}{(x^2+1)^{3/2}} &= \int \frac{\sec^2\theta}{(\tan^2\theta+1)^{3/2}}\, d\theta \\
&= \int \frac{\sec^2\theta}{(\sec^2\theta)^{3/2}}\, d\theta \\
&= \int \frac{\sec^2\theta}{\sec^3\theta}\, d\theta \\
&= \int \frac{1}{\sec\theta}\, d\theta \\
&= \int \cos\theta\, d\theta \\
&= \sin\theta + C.
\end{aligned}$$

Now build a triangle according to the relation $x = \tan\theta$. Then the opposite side will have length x, the adjacent side will have length 1, and the hypoteneuse will have length $\sqrt{1+x^2}$. Then $\sin\theta = \frac{x}{\sqrt{1+x^2}}$. Thus

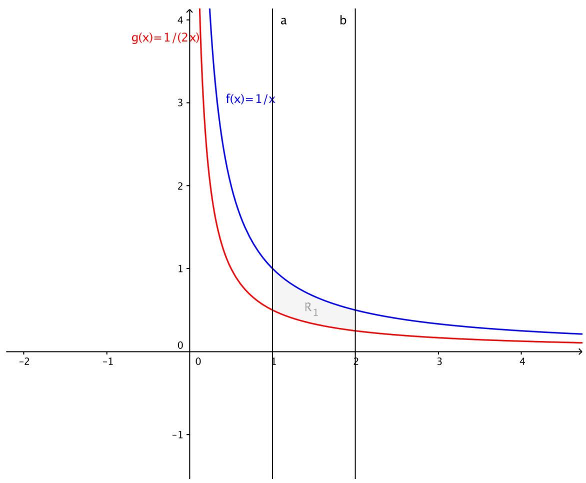
$$\int \frac{dx}{(x^2+1)^{3/2}} = \frac{x}{\sqrt{1+x^2}} + C.$$

2. (a) (6 points) Let R_1 be the region in the first quadrant bounded by

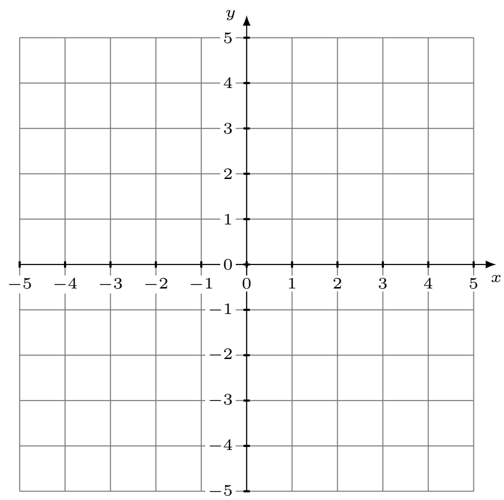
$$f(x) = \frac{1}{x}, \quad g(x) = \frac{1}{2x}, \quad x = 1, \quad \text{and} \quad x = 2.$$

Find the volume of the solid obtained by rotating the region R_1 around the x-axis.

SOLUTION:

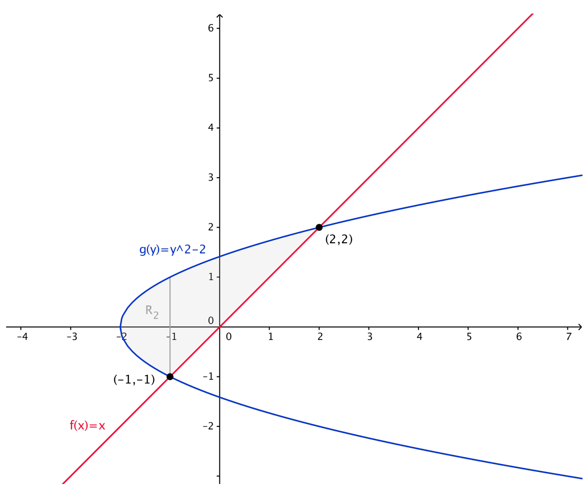


$$\begin{aligned}
\text{Volume} &= \int_1^2 \pi \left(\left(\frac{1}{x}\right)^2 - \left(\frac{1}{2x}\right)^2 \right) dx \\
&= \pi \int_1^2 \frac{1}{x^2} - \frac{1}{4x^2}\, dx \\
&= \pi \int_1^2 \frac{3}{4x^2}\, dx \\
&= \frac{3\pi}{4} \left(\frac{-1}{x} \bigg|_1^2 \right) \\
&= \frac{3\pi}{4} \left(\frac{-1}{2} - (-1) \right) \\
&= \frac{3\pi}{4} \left(\frac{1}{2} \right) = \frac{3\pi}{8}.
\end{aligned}$$



(b) (6 points) Let R_2 be the region enclosed by the line $y = x$ and the curve $x = y^2 - 2$. Draw R_2 on the coordinate axis below and label the points of intersection. Then set up **but do not evaluate** an integral that computes the area of R_2. You may choose to integrate with respect to x or y.

SOLUTION:



With respect to y:

$$\begin{aligned} \text{Area} &= \int_{-1}^{2} y - (y^2 - 2)\,dy \\ &= \int_{-1}^{2} -y^2 + y + 2\,dy \end{aligned}$$

With respect to x:

$$\begin{aligned}\text{Area} &= \int_{-2}^{-1} \sqrt{x+2} - (-\sqrt{x+2})\ dx + \int_{-1}^{2} \sqrt{x+2} - x\,dx \\ &= \int_{-2}^{-1} 2\sqrt{x+2}\,dx + \int_{-1}^{2} \sqrt{x+2} - x\,dx\end{aligned}$$

3. (a) (6 points) Determine whether the series converges or diverges: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 5n}$.

SOLUTION: Use the Comparison Test. For $n \geq 1$,

$$2^n + 5n \geq 2^n.$$

Thus

$$\frac{1}{2^n + 5n} \leq \frac{1}{2^n}$$

for $n \geq 1$. The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges because it is a geometric series with $|r| = \frac{1}{2} < 1$. Then by the Comparison Test, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n + 5n}$ **converges**.

(b) (6 points) Determine whether the series converges or diverges: $\displaystyle\sum_{k=2}^{\infty} (-1)^k \frac{2k}{2k-1}$.

SOLUTION: Apply the Alternating Series Test. Let $c_k = \dfrac{2k}{2k-1}$. Then for $k \geq 2$, $c_k \geq 0$. Going further,

$$\lim_{k \to \infty} \frac{2k}{2k-1} = 1 \neq 0.$$

Then by the Alternating Series Test, the series **diverges**.

(c) (4 points) Does the series in part (b) converge absolutely? Explain why or why not.

SOLUTION: The series in part (b) does **not** converge absolutely. It does not converge conditionally, so it will not converge absolutely. The series

$$\sum_{k=2}^{\infty} \left| (-1)^k \frac{k}{2k-1} \right| = \sum_{k=2}^{\infty} \frac{k}{2k-1}$$

fails the kth term test and thus it **diverges**.

4. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+2)^2}$.

(a) (8 points) Use the Integral Test to determine if the series above converges or diverges. Verify all assumptions of the test.

SOLUTION: Let $f(x) = \dfrac{2x}{(x^2+2)^4}$. First to apply the Integral Test I must check whether $f(x) \geq 0$ for $x \geq 1$ and if $f(x)$ is *decreasing* for $x \geq 1$. If these are true, then the series will converge if and only if $\displaystyle\int_1^{\infty} f(x)\,dx$ converges.

It is clear that for $x \geq 1$, $2x$ and $(x^2+2)^4$ are positive, thus $f(x) \geq 0$.

Next, compute

$$\begin{aligned} f'(x) &= \frac{2(x^2+2)^4 - 2x\big(4(x^2+2)^3 \cdot 2x\big)}{(x^2+2)^8} \\ &= 2(x^2+2)^3\left(\frac{x^2+2-8x^2}{(x^2+2)^8}\right) \\ &= \frac{2(-7x^2+2)}{(x^2+2)^5}. \end{aligned}$$

For $x \geq 1$, $-7x^2+2 \leq -5$, therefore $f'(x) \leq 0$ for $x \geq 1$ and now we know that f is decreasing.

Finally,

$$\begin{aligned} \int_1^{\infty} \frac{2x}{(x^2+2)^4}\,dx &= \lim_{t\to\infty} \int_1^t \frac{2x}{(x^2+2)^4}\,dx \\ &= \lim_{t\to\infty}\left(-\frac{1}{(x^2+2)^3}\Bigg|_1^t\right) \\ &= \lim_{t\to\infty}\left(-\frac{1}{(t^2+2)^3} + \frac{1}{27}\right) \\ &= \frac{1}{27}. \end{aligned}$$

Then the improper integral **converges**.

Thus by the Integral Test, $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+2)^3}$ **converges**.

(b) (4 points) If the series converges, use the Integral Test to find an upper bound for $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{(n^2+2)^4}$. If the series diverges, leave this problem blank.

SOLUTION: The integral test says that if the series converges, then

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2n}{(n^2+2)^4} &\leq c_1 + \int_1^{\infty} \frac{2x}{(x^2+2)^4}\,dx \\ &= \frac{2}{3^4} + \frac{1}{27} \\ &= \frac{5}{81} = 0.0617.\end{aligned}$$

5. Consider the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{3^n}(x-1)^n$.

(a) (6 points) Find the radius of convergence of the power series. Clearly indicate which tests you use.

SOLUTION: Use the Ratio Test to determine the radius of convergence.

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{(n+1)^2(x-1)^{n+1}}{3^{n+1}}\right|\cdot\left|\frac{3^n}{n^2(x-1)^n}\right| &= \lim_{n\to\infty}\frac{(n+1)^2}{n^2}\cdot\frac{|x-1|}{3} \\ &= \frac{|x-1|}{3}.\end{aligned}$$

The series converges absolutely when the ratio has limit strictly less than 1, in this case whenever $\dfrac{|x-1|}{3} < 1$. Then the series converges absolutely by the Ratio Test when $|x-1| < 3$. The radius of convergence is 3.

(b) (6 points) Find the interval of convergence of the power series. Clearly indicate which tests you use, and verify that all of the necessary assumptions are satisfied.

SOLUTION: By part (a), the series converges absolutely when $|x-1| < 3$, or equivalently when $-2 < x < 4$. Next test the endpoints $x = -2$ and $x = 4$. At $x = -2$ the power series becomes the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n n^2 (-3)^n}{3^n} = \sum_{n=1}^{\infty} n^2.$$

This series **diverges** by the nth term test since $\lim_{n\to\infty} n^2 \neq 0$.

At $x = 4$ the power series becomes

$$\sum_{n=1}^{\infty} (-1)^n n^2.$$

This series **diverges** by the nth term test as well. Here, $\lim_{n\to\infty}(-1)^n n^2$ does not exist.

Thus the interval of convergence is $-2 < x < 4$.

6. Let $f(x) = \dfrac{1}{4x^2 - 1}$ and let N be a natural number ($N = 1, 2, 3, \dots$).

(a) (4 points) Put in increasing order the quantities $I = \displaystyle\int_1^{N+1} f(x)\,dx$, L_N, and R_N.

SOLUTION: The function $f(x)$ is decreasing for $x \geq 1$, therefore

$$R_N \leq I \leq f_N.$$

(b) (4 points) Consider $I = \displaystyle\int_1^{N+1} f(x)\,dx$. The sum $\displaystyle\sum_{n=1}^{N} \frac{1}{4n^2 - 1}$ is equal to one of L_N or R_N. Determine the correct choice and explain your answer.

SOLUTION:

$$L_N = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

To compute L_N for I, break the interval $[1, N+1]$ into N equal pieces - so each will have length 1. Then evaluate $f(x)$ at the left-side of each of the subintervals. This will mean that

$$L_N = 1\,(f(1) + f(2) + f(3) + \cdots + f(N)) = \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1}.$$

(c) (6 points) Find a simple expression for $\sum_{n=1}^{N} \frac{1}{4n^2-1}$. *Hint: use partial fractions.*

SOLUTION: Find a partial fraction decomposition for

$$\frac{1}{4n^2-1} = \frac{A}{2n-1} + \frac{B}{2n+1}.$$

Then we have that $A(2n+1) + B(2n-1) = 1$, so $2A + 2B = 0$ and $A - B = 1$. This system of equations has solution $A = 1/2$ and $B = -1/2$. Then

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{4n^1-1} &= \sum_{n=1}^{N} \left(\frac{1/2}{2n-1} - \frac{1/2}{2n+1} \right) \\ &= \frac{1}{2} \sum_{n=1}^{N} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \\ &= \frac{1}{2} \left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2N-1} - \frac{1}{2N+1} \right) \right] \\ &= \frac{1}{2} \left(1 - \frac{1}{2N+1} \right). \end{aligned}$$

(d) (2 points) Find $\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{4n^2-1}$.

SOLUTION:

$$\begin{aligned}\lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{4n^2-1} &= \lim_{N\to\infty} \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) \\ &= \frac{1}{2}.\end{aligned}$$

Bonus. (4 points) Give an explanation why $\int_1^\infty \frac{1}{4x^2-1}\,dx \leq \lim_{N\to\infty} \sum_{n=1}^{N} \frac{1}{4n^2-1}$.

SOLUTION: The function $f(x)$ is decreasing for $x \geq 1$, therefore left-hand rectangles always give an over-estimate of $\int_1^\infty f(x)\,dx$. The quantity on the right is just the limit as the number of rectangles go to infinity of L_N.

7. (a) (6 points) What is $\sum_{k=0}^{\infty} 2^{-2k}$?

SOLUTION: Rewrite the term

$$a_k = 2^{-2k} = (2^{-2})^k = \left(\frac{1}{4}\right)^k.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty} 2^{-2k} &= \sum_{k=0}^{\infty} \left(\frac{1}{4}\right)^k \\ &= \frac{1}{1-\frac{1}{4}} \\ &= \frac{4}{3}.\end{aligned}$$

(b) (6 points) Find the first four non-zero terms of the Maclaurin series for

$$f(x) = x\sin(2x).$$

SOLUTION:

$$\begin{aligned}x\big(\sin(2x)\big) &= x\left((2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \dots\right) \\ &= x\left(2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \dots\right) \\ &= 2x^2 - \frac{8x^4}{3!} + \frac{32x^6}{4!} - \frac{128x^8}{7!} + \dots.\end{aligned}$$

(c) (2 points) Find $f^{(2016)}(0)$ exactly for $f(x) = x\sin(2x)$.

SOLUTION: By the formula for Taylor series,

$$f^{(2016)}(0) = a_{2016} \cdot (2016!).$$

Now by part (b) I can see that $a_{2016} = \frac{2^{2015}}{2015!}$. Then

$$f^{(2016)}(0) = \frac{2^{2015}}{2015!} \cdot (2016!) = \frac{2^{2015}}{2016}.$$