Math 106C Name ____________________

Final Exam Version 1

12/16/15

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable. Give exact answers unless otherwise noted.

1. (20 pts.) Evaluate the following integrals.

   (a) $\int e^{2x} e^{x} \, dx$

   (b) $\int \sin^2 x \cos^3 x \, dx$

   (c) $\int \arctan x \, dx$

(d) $\displaystyle\int \frac{x}{(1+x^2)^{3/2}}\,dx$

2. (8 pts.) The Maclaurin series for $\ln(1+t) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}t^k}{k}$ on $(-1,1)$. Find the first four non-zero terms of the Maclaurin series for $f(x) = \dfrac{\ln(1+2x)}{x}$.

3. (8 pts.) Find the interval of convergence for the series $\displaystyle\sum_{k=1}^{\infty} \frac{3(2x-3)^k}{k \cdot 7^k}$.

4. (8 pts.) Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x^2+2}{3y^2}$ with $y(0) = 2$.

5. (8 pts.) Does $\sum_{k=2}^{\infty} \frac{\ln k}{k}$ converge or diverge? Justify your answers and identify any convergence test that you use. If the series converges then find an exact value for both an upper and lower bound for the series.

6. (8 pts.) Each of the following statements is an attempt to show that a given series is convergent or divergent using the Comparison Test. For each statement, enter C (for "*correct*") if the argument is valid, or enter I (for "*incorrect*") if *any* part of the argument is flawed. **(Note: if the conclusion is true but the argument that led to it was wrong, you must enter I.)**

   (a) ______ For all $k > 2$: $\frac{\ln k}{k^2} > \frac{1}{k^2} > 0$, and the series $\sum_{k=3}^{\infty} \frac{1}{k^2}$ converges. Therefore, by the Comparison Test the series $\sum_{k=3}^{\infty} \frac{\ln k}{k^2}$ converges.

   (b) ______ For all $k \geq 1$: $0 < \frac{\arctan k}{k^3} < \frac{\pi}{2k^3}$, and the series $\frac{\pi}{2} \sum_{k=1}^{\infty} \frac{1}{k^3}$ converges. Therefore, by the Comparison Test the series $\sum_{k=1}^{\infty} \frac{\arctan k}{k^3}$ converges.

   (c) ______ For all $k > 1$: $\frac{k}{4 - k^3} < \frac{1}{k^2}$, and the series $\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges. Therefore, by the Comparison Test the series $\sum_{k=3}^{\infty} \frac{k}{4 - k^3}$ converges.

   (d) ______ For all $k \geq 1$: $\frac{3}{7^k} < \frac{4 + \cos k}{7^k} < \frac{5}{7^k}$, and the geometric series $\sum_{k=1}^{\infty} \frac{5}{7^k}$ converges. Therefore, by the Comparison Test the series $\sum_{k=1}^{\infty} \frac{4 + \cos k}{7^k}$ converges.

7. (8 pts.) **True/False.** Determine if each of the following statements are true or false. If a statement is false, write a sentence or two explaining why the statement is false or give an example to demonstrate why.

(a) The geometric series $1 + \arctan 1 + (\arctan 1)^2 + (\arctan 1)^3 + (\arctan 1)^4 + \cdots$ diverges.

(b) If the terms of a series alternate in sign, decrease in magnitude, and approach zero, then the series converges.

(c) The series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(k-1)}$ converges because $\displaystyle\lim_{k\to\infty} \frac{1}{k(k-1)} = 0$.

(d) If $0 \leq f(x) \leq \dfrac{1}{x^{3/2}}$ for $k \geq 1$, then $\displaystyle\int_{k=1}^{\infty} f(x)\,dx$ converges.

(e) The interval of convergence for the power series $\displaystyle\sum_{k=1}^{\infty} kx^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$ is $-1 < x < 1$, therefore the radius of convergence is 1.

(f) If $|r| < 1$, then $\displaystyle\sum_{k=1}^{\infty} ar^k = \frac{a}{1-r}$.

(g) The p-test for series states that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p \geq 1$.

(h) The series $1 + \dfrac{1}{4} + \dfrac{1}{2!}\left(\dfrac{1}{4}\right)^2 + \dfrac{1}{3!}\left(\dfrac{1}{4}\right)^3 + \dfrac{1}{4!}\left(\dfrac{1}{4}\right)^4 + \cdots$ converges to e^{-4}.

Multiple Choice (24 pts. total – 3 pts. each)

8. Recall that a power series has the form $\sum_{k=0}^{\infty} a_k(x-c)^k = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$.
Which of the following infinite series are power series?
SELECT ALL THAT APPLY.

(a) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^k}{k!} + \cdots$

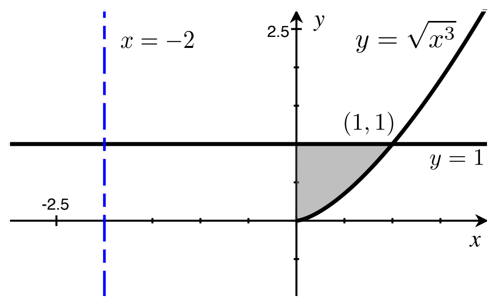
(b) $\sum_{k=0}^{\infty}(x-5)^k$

(c) $1 - \frac{1}{x} + \frac{2!}{x^2} - \frac{3!}{x^3} + \cdots + \frac{(-1)^{k+1}k!}{x^k} + \cdots$

(d) $1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{k!} + \cdots$

(e) $\frac{(x+1)}{\sqrt{2}} + \frac{(x+1)^2}{\sqrt{4}} + \frac{(x+1)^3}{\sqrt{8}} + \frac{(x+1)^4}{\sqrt{16}} + \cdots$

9. Consider the region bounded by $y = \sqrt{x^3}$, $y = 1$, and the y-axis. Which of the following represents an integral expression for the volume of the solid created when region described above is rotated around the line $x = -2$?



(a) $\pi \int_0^1 \left[\left(2 + \sqrt[3]{y^2}\right)^2 - 4\right] dy$

(b) $\pi \int_0^1 \left[\left(2 + \sqrt[3]{y^2}\right) - 2\right]^2 dy$

(c) $\pi \int_0^1 \left[\left(\sqrt{x^3} + 2\right)^2 - 4\right] dx$

(d) $\pi \int_0^1 \left[4 - \left(\sqrt{x^3} + 2\right)^2\right] dx$

10. The power series $\sum_{k=1}^{\infty} kx^{k-1} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$ converges only for x values in the interval from $(-1, 1)$. If $x = 1$, the series becomes $1 + 2(1) + 3(1)^2 + 4(1)^3 + 5(1)^4 + \cdots = \sum_{k=1}^{\infty} k$. Which of the following statements is true? **SELECT ALL THAT APPLY.**

(a) The series $\sum_{k=1}^{\infty} k$ diverges by the k^{th}-term test because $\lim_{k\to\infty} k = \infty \neq 0$.

(b) The power series $\sum_{k=1}^{\infty} kx^{k-1}$ diverges when $x = 1$.

(c) The series $\sum_{k=1}^{\infty} k$ diverges by the alternating series test.

(d) The series $\sum_{k=1}^{\infty} k$ diverges by the ratio test.

11. Consider the power series given by $\sum_{n=2}^{\infty} \frac{(x+5)^n}{n \ln n}$. Which of the following expressions represents the series when $x = -6$?

(a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$

(b) $\sum_{n=2}^{\infty} \frac{(-6)^n}{n \ln n}$

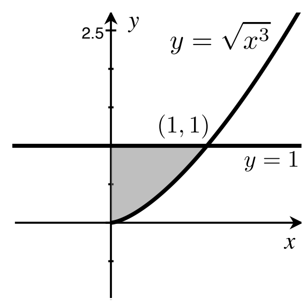
(c) $\sum_{n=2}^{\infty} \frac{(x+5)^{-6}}{n \ln n}$

(d) $\sum_{n=2}^{\infty} \frac{(-6)(x+5)^n}{n \ln n}$

12. Which trigonometric substitution is needed to evaluate the integral $\int \frac{x^3}{\sqrt{9-x^2}}\,dx$?

(a) $x = \sin\theta$
(b) $x = 9\sec\theta$
(c) $x = 3\sin\theta$
(d) $x = 3\sec\theta$
(e) $x = 9\sin\theta$

13. Consider the region bounded by $y = \sqrt{x^3}$, $y = 1$, and the y-axis. Which of the following represents an integral expression for the volume of the solid created when region described above is rotated around the line $y = 1$?



(a) $\pi \int_0^1 \left(\sqrt[3]{y^2}\right)^2 dy$

(b) $\pi \int_0^1 \left(1 - (\sqrt[3]{y^2})^2\right) dy$

(c) $\pi \int_0^1 \left(1 - \sqrt{x^3}\right)^2 dx$

(d) $\pi \int_0^1 \left(\sqrt{x^3} - 1\right)^2 dx$

(e) $\pi \int_0^1 \left(\sqrt{x^3}\right)^2 dx$

14. Suppose that the fourth-order Maclaurin polynomial for a function f is given by

$$P_4(x) = 3 + \frac{9x}{2} + \frac{9x^2}{2} + \frac{27x^3}{8} + \frac{81x^4}{40}.$$

What is $f'''(0)$?

(a) $\frac{27}{8}$

(b) $\frac{27}{3!}$

(c) $\frac{1}{3!}$

(d) $\frac{81}{4}$

15. Which of the following is the correct form for the partial fraction decomposition of $\dfrac{x^2 - x}{(x+1)^2(x^2+5)}$?

(a) $\dfrac{Ax+B}{x^2+5} + \dfrac{Cx+D}{(x+1)^2}$

(b) $\dfrac{Ax+B}{x^2+5} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$

(c) $\dfrac{A}{x^2+5} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

(d) $\dfrac{Ax+B}{x^2+5} + \dfrac{C}{x+1} + \dfrac{Dx+E}{(x+1)^2}$

(e) $\dfrac{A}{x^2+5} + \dfrac{B}{(x+1)^2}$

16. (8 pts.) Measurements are made of the rate at which water is draining from a container at various times, and recorded in the following table.

t (min)	0	0.5	1	1.5	2
$r(t)$ (liters/min)	17	16	14	10	2

(a) In a sentence or two, explain why someone would want to calculate $\int_0^2 r(t)\,dt$. What is the meaning of this integral in the context of the problem?

(b) Which of the *left*, *right*, *midpoint*, and *trapezoid* sums should you choose to produce the best approximation you can to the value of $\int_0^2 r(t)\,dt$. Calculate the sum you selected, be sure to show enough work so that your method is clear. Provide a sentence to explain why you think this is the best approximation given the limitations of the table.

EXTRA CREDIT

Use *long division* and the Maclaurin series for $\sin x$ and $\cos x$ to find the first 3 non-zero terms of Maclaurin series for $f(x) = \tan x$.

Formulae you may find useful.

- $\int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$
- $\int \csc\theta \, d\theta = \ln|\csc\theta - \cot\theta| + C$
- $\int \cot\theta \, d\theta = -\ln|\csc\theta| + C$
- $\int_a^b \sqrt{1 + (f'(x))^2} \, dx$
- $P_n(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$
- $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$
- $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$
- Log Properties
    - $\ln(xy) = \ln x + \ln y$
    - $\ln \frac{x}{y} = \ln x - \ln y$
    - $\ln x^y = y \ln x$
- Trigonometric Identities
    - $\sin^2\theta + \cos^2\theta = 1$
    - $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
    - $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
    - $\sec^2\theta = 1 + \tan^2\theta$
    - $\sin(2\theta) = 2\sin\theta\cos\theta$
    - $\cos(2\theta) = cos^2\theta - \sin^2\theta$