

**Math 106: Review for Final Exam, Part II - SOLUTIONS**

1. Use a second-order Taylor polynomial to estimate  $\sqrt[3]{28}$ .

We choose  $f(x) = \sqrt[3]{x}$  and  $x_0 = 27$  because 27 is the perfect cube closest to 28.

$$\begin{aligned} f(x) &= x^{1/3} & f(27) &= 3 \\ f'(x) &= \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}} & f'(27) &= \frac{1}{3 \cdot 27^{2/3}} = \frac{1}{27} \\ f''(x) &= -\frac{2}{9}x^{-5/3} = -\frac{2}{9x^{5/3}} & f''(27) &= -\frac{2}{9 \cdot 27^{5/3}} = -\frac{2}{2187} \end{aligned}$$

Now plug in to the Taylor polynomial formula with  $x_0 = 27$ .

$$P_2(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 = 3 + \frac{1}{27}(x - 27) - \frac{1}{2187}(x - 27)^2$$

Finally, evaluate at  $x = 28$ .

$$\sqrt[3]{28} \approx P_2(28) = 3 + \frac{1}{27}(28 - 27) - \frac{1}{2187}(28 - 27)^2 = \frac{6641}{2187} \approx 3.0365797$$

2. What is the largest possible error that could have occurred in your previous estimate?

We know that  $|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$ .

In this case,  $n = 2$ ,  $x_0 = 27$ , and  $x = 28$ .

$$K_3 = \max \text{ of } |f'''(x)| \text{ on } [27, 28] = \max \text{ of } \left| \frac{10}{27x^{8/3}} \right| \text{ on } [27, 28] = \frac{10}{27 \cdot 27^{8/3}} = \frac{10}{177147}$$

Putting this all together, we have  $|f(x) - P_2(x)| \leq \frac{10}{3!} \frac{1}{177147} |28 - 27|^3 = \frac{5}{531441} \approx 0.0000094$ .

3. Use a comparison to show whether each of the following converges or diverges. If an integral converges, give a good upper bound for its value.

(a)  $\int_1^{\infty} \frac{7 + 5 \sin x}{x^2} dx$

For all  $x \geq 1$ , we have  $0 \leq \frac{7 + 5 \sin x}{x^2} \leq \frac{7 + 5(1)}{x^2} = 12 \frac{1}{x^2}$  because the maximum of  $\sin x$  is 1.

$$\begin{aligned} 12 \int_1^{\infty} \frac{dx}{x^2} &= 12 \lim_{t \rightarrow \infty} \int_1^t \frac{dx}{x^2} \\ &= 12 \lim_{t \rightarrow \infty} \left. \frac{-1}{x} \right|_1^t \\ &= 12 \lim_{t \rightarrow \infty} \left[ \frac{-1}{t} - \frac{-1}{1} \right] \\ &= 12[0 - (-1)] \\ &= 12 \end{aligned}$$

Therefore, the original integral in question must converge to a value less than 12.

(b)  $\int_1^{\infty} \frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}} dx$

For  $x \geq 1$ , we have  $\frac{1 + 3x^2 + 2x^3}{\sqrt[3]{10x^{12} + 17x^{10}}} \geq \frac{2x^3}{\sqrt[3]{10x^{12} + 17x^{12}}} \geq 0$ . (We've made the numerator smaller and the denominator larger, so the new fraction is smaller.)

But  $\frac{2x^3}{\sqrt[3]{10x^{12} + 17x^{12}}} = \frac{2x^3}{\sqrt[3]{27x^{12}}} = \frac{2x^3}{3x^4} = \frac{2}{3} \frac{1}{x}$  and we know that  $\frac{2}{3} \int_1^\infty \frac{dx}{x}$  diverges (compute for yourself or notice that  $p = 1$ ).

Therefore the original integral must also diverge.

4. Decide if each of the following sequences  $\{a_k\}_{k=1}^\infty$  converges or diverges. If a sequence converges, compute its limit.

(a)  $a_k = 3 + \frac{1}{10^k}$  Terms are 3.1, 3.01, 3.001, 3.0001, ...

$\lim_{k \rightarrow \infty} \left( 3 + \frac{1}{10^k} \right) = 3$ , so the sequence converges to 3.

(b)  $a_k = (-1)^k$  Terms are  $-1, 1, -1, 1, \dots$

$\lim_{k \rightarrow \infty} (-1)^k$  doesn't exist, so the sequence diverges.

(c)  $a_k = \frac{5e^k}{7e^k + \ln(k+1)}$  Terms are  $\frac{5e}{7e + \ln 2}, \frac{5e^2}{7e^2 + \ln 3}, \frac{5e^3}{7e^3 + \ln 4}, \dots$

By L'Hopital's Rule,  $\lim_{k \rightarrow \infty} \frac{5e^k}{7e^k + \ln(k+1)} = \lim_{k \rightarrow \infty} \frac{5e^k}{7e^k + \frac{1}{k+1}} = \frac{5}{7}$ , so the sequence converges to

$\frac{5}{7}$ .

5. Decide if each of the following series converges or diverges. If a series converges, find its value.

(a)  $3.1 + 3.01 + 3.001 + 3.0001 + \dots$

$\lim_{k \rightarrow \infty} a_k = 3 \neq 0$ , so the series diverges by the nth Term Test. (We keep adding 3's forever.)  
[Compare this with the first sequence of the previous problem.]

(b)  $1 + 1/2 + 1/3 + 1/4 + \dots$

This is the famous Harmonic Series, which diverges *even though* the terms do approach 0. We can use the Integral Test:  $\int_1^\infty \frac{1}{x} dx$  diverges, which means that  $\sum_{k=1}^\infty \frac{1}{k}$  must diverge too.

(c)  $5 - 5/3 + 5/9 - 5/27 + \dots$

This is a geometric series with  $r = -\frac{1}{3}$ , so it converges to  $\frac{a}{1-r} = \frac{5}{1 - (-1/3)} = \frac{15}{4}$ .

6. Decide if each of the following series converges or diverges. If a series converges, find upper and lower bounds for its value.

(a)  $\sum_{k=1}^\infty \frac{(-1)^k}{\sqrt[3]{k+1}}$  [Alternating Series Test]

The terms of this series alternate in sign.

And,  $\frac{1}{\sqrt[3]{2}} \geq \frac{1}{\sqrt[3]{3}} \geq \frac{1}{\sqrt[3]{4}} \geq \dots \geq 0$ . (Or, more formally,  $k < k+1 \Rightarrow (k+1) < (k+1)+1 \Rightarrow \sqrt[3]{k+1} < \sqrt[3]{(k+1)+1} \Rightarrow \frac{1}{\sqrt[3]{k+1}} > \frac{1}{\sqrt[3]{(k+1)+1}}$  so the terms are decreasing in size.)

And,  $\lim_{k \rightarrow \infty} \frac{1}{\sqrt[3]{k+1}} = 0$ .

Therefore, by the Alternating Series Test, the series must converge.

We know that any two consecutive partial sums will provide upper and lower bounds:

lower bound =  $S_1 = \frac{-1}{\sqrt[3]{2}}$       upper bound =  $S_2 = \frac{-1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}}$

[To get better bounds, use later partial sums, such as  $S_5$  and  $S_6$ .]

(b)  $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k(k!)^2}$  [Ratio Test]

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{a_{k+1}}{a_k} &= \frac{\frac{[2(k+1)]!}{3^{k+1}[(k+1)!]^2}}{\frac{(2k)!}{3^k(k!)^2}} \\ &= \lim_{k \rightarrow \infty} \frac{(2k+2)!}{(2k)!} \frac{3^k}{3^{k+1}} \frac{(k!)^2}{[(k+1)!]^2} \\ &= \lim_{k \rightarrow \infty} \frac{(2k+2)(2k+1)}{1} \frac{1}{3} \frac{1}{(k+1)^2} \\ &= \frac{4k^2 + 6k + 2}{3k^2 + 6k + 3} \quad \text{Use L'Hopital or divide each term by } k^2. \\ &= \frac{4}{3} \end{aligned}$$

Since the limit of the ratio is greater than 1, the series diverges.

(c)  $\sum_{k=1}^{\infty} \left( \frac{1}{100} + \frac{1}{k^5} \right)$  [nth Term Test]

$$\lim_{k \rightarrow \infty} \left( \frac{1}{100} + \frac{1}{k^5} \right) = \frac{1}{100} \neq 0, \text{ so, by the nth Term Test, the series diverges.}$$

(d)  $\sum_{k=1}^{\infty} \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3}$  [Comparison Test]

$$\text{For } k \geq 1, \text{ we have } \frac{\sqrt{9k^8 + 5k^6}}{12k^5 + 3} > \frac{\sqrt{9k^8}}{12k^5 + 3k^5} > 0.$$

But,  $\frac{\sqrt{9k^8}}{12k^5 + 3k^5} = \frac{3k^4}{15k^5} = \frac{1}{5} \frac{1}{k}$  and we know that  $\sum_{k=1}^{\infty} \frac{1}{k}$  diverges (use Integral Test or note that  $p = 1$ ).

Therefore, the original series, which has larger terms, must diverge also.

(e)  $\sum_{k=2}^{\infty} \frac{1}{k(\ln(k))^2}$  [Integral Test]

$$\begin{aligned} \int_2^{\infty} \frac{1}{x(\ln x)^2} dx &= \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln x)^2} dx \\ &= \lim_{t \rightarrow \infty} \int_{x=2}^{x=t} \frac{1}{u^2} du \quad \text{Substitute } u = \ln x, \text{ so } du = \frac{dx}{x}. \\ &= \lim_{t \rightarrow \infty} \frac{u^{-1}}{-1} \Big|_{x=2}^{x=t} \\ &= \lim_{t \rightarrow \infty} \frac{-1}{\ln x} \Big|_2^t \\ &= \lim_{t \rightarrow \infty} \left[ \frac{-1}{\ln t} - \frac{-1}{\ln 2} \right] \\ &= 0 - \frac{-1}{\ln 2} \\ &= \frac{1}{\ln 2} \end{aligned}$$

The integral converges, so the series must converge too.

Further, we know that  $\int_2^\infty \frac{1}{x(\ln x)^2} dx \leq \sum_{k=2}^\infty \frac{1}{k(\ln(k))^2} \leq a_2 + \int_2^\infty \frac{1}{x(\ln x)^2} dx$ .

Therefore, our lower bound is  $\int_2^\infty \frac{1}{x(\ln x)^2} dx = \frac{1}{\ln 2}$ .

And our upper bound is  $a_2 + \int_2^\infty \frac{1}{x(\ln x)^2} dx = \frac{1}{2(\ln 2)^2} + \frac{1}{\ln 2}$ .

7. Does the first series from the previous problem converge absolutely or conditionally?

$\sum_{k=1}^\infty \left| \frac{(-1)^k}{\sqrt[3]{k+1}} \right| = \sum_{k=1}^\infty \frac{1}{\sqrt[3]{k+1}}$ , which diverges by the Integral Test (check for yourself).

Therefore, the first series from the previous problem converges conditionally.

8. Compute the radius and interval (including endpoints) of convergence for  $\sum_{k=1}^\infty \frac{(x+3)^k}{k \cdot 5^k}$ .

$$\begin{aligned} \lim_{k \rightarrow \infty} \left| \frac{\frac{(x+3)^{k+1}}{(k+1) \cdot 5^{k+1}}}{\frac{(x+3)^k}{k \cdot 5^k}} \right| &= \lim_{k \rightarrow \infty} \left| \frac{(x+3)^{k+1}}{(x+3)^k} \frac{k}{k+1} \frac{5^k}{5^{k+1}} \right| && \text{Use L'Hopital on the middle fraction.} \\ &= \left| (x+3) \cdot 1 \cdot \frac{1}{5} \right| \\ &= \left| \frac{x+3}{5} \right| \end{aligned}$$

So, we are guaranteed convergence when  $\left| \frac{x+3}{5} \right| < 1$ . But this is equivalent to the following.

$$\begin{aligned} -1 &< \frac{x+3}{5} < 1 \\ -5 &< x+3 < 5 \\ -8 &< x < 2 \end{aligned}$$

To check convergence at the endpoints (where the Ratio Test is inconclusive), we plug in to the series itself.

At  $x = 2$ , we have  $\sum_{k=1}^\infty \frac{(2+3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{5^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{1}{k}$ , which is the Harmonic Series and thus diverges.

At  $x = -8$ , we have  $\sum_{k=1}^\infty \frac{(-8+3)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-5)^k}{k \cdot 5^k} = \sum_{k=1}^\infty \frac{(-1)^k}{k}$ , which converges by the Alternating Series Test.

Thus, the interval of convergence is  $-8 \leq x < 2$  and the radius of convergence is 5.

9. Evaluate the following exactly.

(a)  $1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \dots$

We recall that  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots$ .

The series in question is the series for  $e^x$  with  $x = -1$ ; therefore, it converges to  $e^{-1}$ .

(b)  $1 - \frac{\pi^2}{2} + \frac{\pi^4}{24} - \frac{\pi^6}{720} + \dots$

We recall that  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \dots$ .

The series in question is the series for  $\cos x$  with  $x = \pi$ ; therefore, it converges to  $\cos \pi$ , which is  $-1$ .

10. (a) **Using summation notation, write the series equal to  $\int_0^1 e^{-x^2} dx$  and show that it converges.**

We know  $e^w = 1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \dots$ , so by substitution we obtain the following.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

$$\begin{aligned} \text{Thus, } \int_0^1 e^{-x^2} dx &= \int_0^1 \left[ 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots \right] dx \\ &= \left[ x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \right]_0^1 \\ &= 1 - \frac{1^3}{3} + \frac{1^5}{5 \cdot 2!} - \frac{1^7}{7 \cdot 3!} + \dots \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)k!} \end{aligned}$$

The terms of this series alternate in sign.

And,  $1 \geq \frac{1^3}{3} \geq \frac{1^5}{5 \cdot 2!} \geq \frac{1^7}{7 \cdot 3!} \geq \dots \geq 0$ . (Or more formally,  $k < k+1 \Rightarrow 2k+1 < 2(k+1)+1 \Rightarrow \frac{1}{2k+1} > \frac{1}{2(k+1)+1} \Rightarrow \frac{1}{(2k+1)k!} > \frac{1}{(2(k+1)+1)(k+1)!}$ , so the terms are decreasing in size.)

And,  $\lim_{k \rightarrow \infty} \frac{1}{(2k+1)k!} = 0$ .

Therefore, by the Alternating Series Test, the series must converge.

- (b) **If  $f(x) = e^{-x^2}$ , what is  $f^{(400)}(0)$ ? What is  $f^{(401)}(0)$ ?**

In the previous part, we found the following.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \dots = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

We know that  $f^{(400)}(0)$  will appear in the Taylor series for  $f(x)$  in the coefficient of the  $x^{400}$  term, which is  $\frac{f^{(400)}(0)}{400!} x^{400}$ .

From above, we see that term is  $\frac{x^{400}}{200!}$ .

Setting them equal gives  $\frac{f^{(400)}(0)}{400!} x^{400} = \frac{x^{400}}{200!}$ , which means that  $f^{(400)}(0) = \frac{400!}{200!}$ .

[Note that this does *not* simplify to  $200!$ . Rather,  $\frac{400!}{200!} = 400 \cdot 399 \cdot \dots \cdot 202 \cdot 201$ .]

We know that  $f^{(401)}(0)$  will appear in the Taylor series for  $f(x)$  in the coefficient of the  $x^{401}$  term, which is  $\frac{f^{(401)}(0)}{401!} x^{401}$ .

However, the Taylor series above for  $f(x)$  has no terms with odd powers of  $x$ , meaning that the coefficients of all those terms must be 0.

Therefore, we know that  $f^{(401)}(0) = 0$ .