Name: Solutions

Math 206: Fall 2013
Final Exam

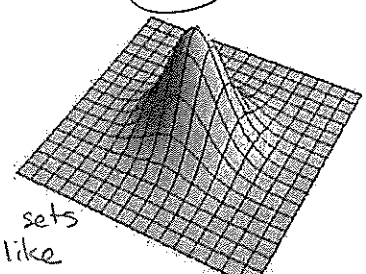
Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable.

Good Luck!

1. (26 points) Solve the following problems. Write your answers on this exam. For all parts of question 1 only, no justification necessary, no partial credit available.

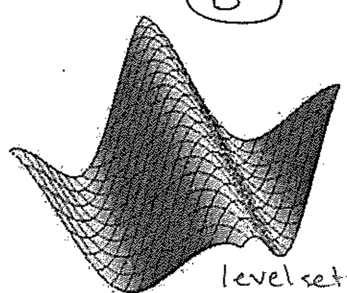
   (a) Match each sketch with an equation.

   (I) A

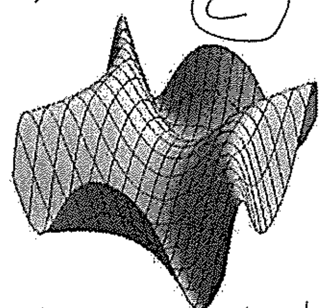
   level sets look like

   (II) B

   level sets look like

   (III) C

   level sets look like

   (A) $f(x, y) = e^{-x^2-y^2}$

   (B) $f(x, y) = \cos(x + y)$

   (C) $f(x, y) = \sin(x^2 - y^2)$

   (D) $f(x, y) = x^4 + y^4$ similar to a paraboloid

   $\sin(x^2-y^2)=0$

   $x^2-y^2 = \pi k$ for all $k \in \mathbb{Z}$

   these are hyperbolas

   $\cos(x+y)=0$

   $x+y = \frac{2k+1}{2}\pi$ for all $k\in\mathbb{Z}$ these are lines

   (b) Let R be the two dimensional region shown in the figure below. What is $\int_R f(x, y)\, dA$?

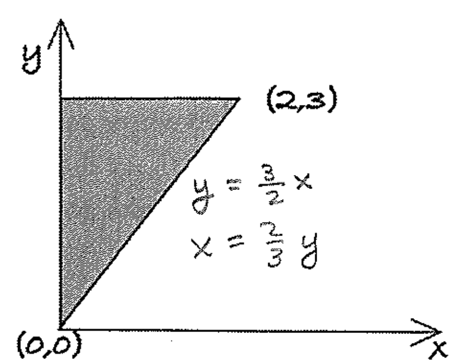
   i. $\int_0^3 \int_0^{2y/3} f(x, y)\, dxdy$

   ~~ii.~~ $\int_0^3 \int_0^{3x/2} f(x, y)\, dxdy$

   ~~iii.~~ $\int_0^3 \int_0^{2y/3} f(x, y)\, dydx$

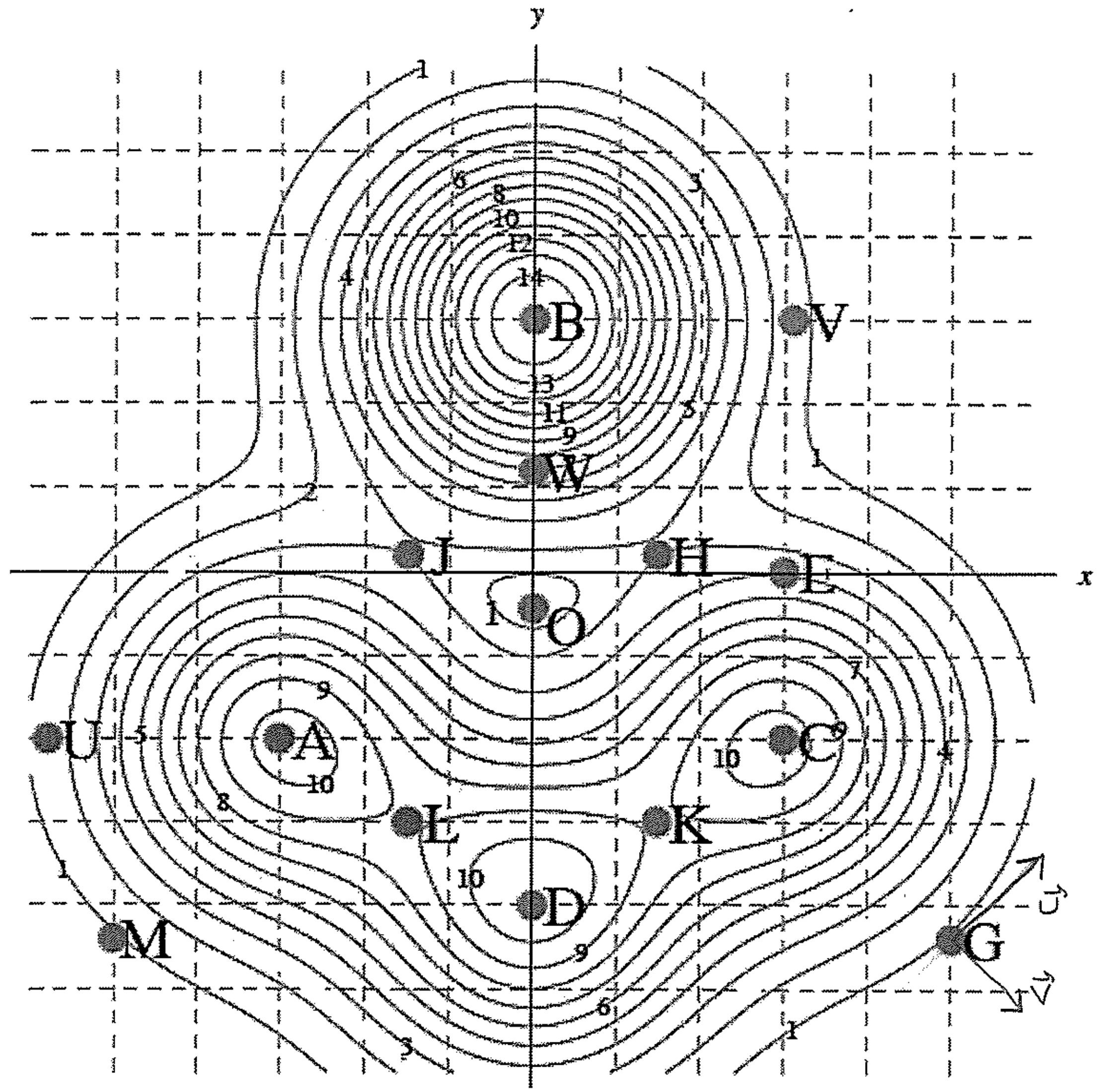
   iv. $\int_0^2 \int_0^{3x/2} f(x, y)\, dydx$

   ~~v.~~ $\int_0^3 \int_{3x/2}^{2} f(x, y)\, dxdy$



ii, iii, v can be ruled out by considering the order of dxdy and the limits of integration

(c) The diagram below depicts contours of a function $f(x, y)$.



Enter one label into each of the boxes.

U At which point is $f_x > 0, f_y = 0$? looking for a point such that the level set through that point is tangent to a vertical line.

G At which point is $f_{\vec{u}} = 0$ and $f_{\vec{v}} < 0$, where $\vec{u} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$ and $\vec{v} = \frac{1}{\sqrt{2}}(\hat{\imath} - \hat{\jmath})$?
looking for a point such that the level set through that point is tangent to a line w/ slope = 1.

B At which point does f have a global maximum?

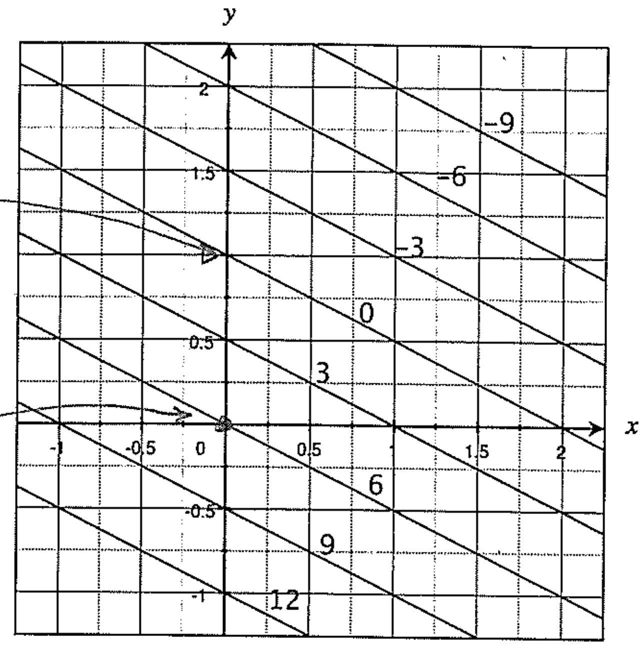
O At which point does f have a local minimum?

E At which point is f maximal under the constraint $y = 0$? find a point on the line $y=0$, at which the line $y=0$ is tangent to the level set through that point

W At which point is the length of the gradient maximal?

where are contours closest together? I.e. where is the rate of change perpendicular to the level set largest?

(d) The figure below shows the contour diagram of which function $f(x,y)$?



i. $f(x,y) = 6y - 3x + 6$

ii. $f(x,y) = \frac{1-x}{2}$

iii. $f(x,y) = e^{-3x-6y+6}$

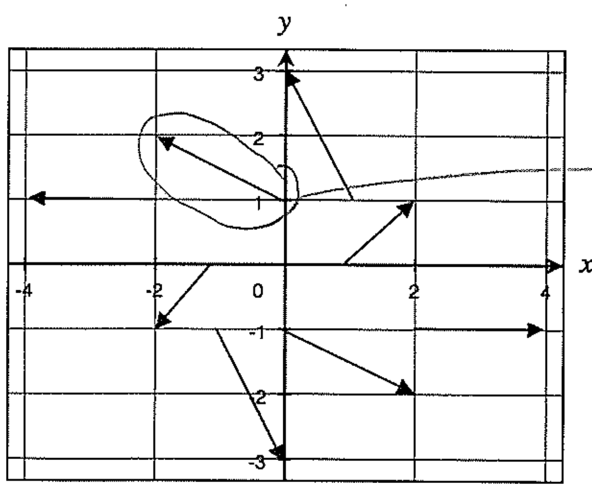
iv. $f(x,y) = -3x - 6y$

v. None of the above.

$(0,1,0)$
$\Rightarrow$ not i, ii, iii, iv

$(0,0,6)$
$\Rightarrow$ NOT ii, iii, iv.

(e) A portion of a vector field $\vec{F}$ is shown below. Find $\vec{F}(0,1)$.



$\boxed{-2\hat{\imath} + \hat{\jmath}}$

(f) Suppose that $f(x,y)$ has continuous first and second order partial derivatives at $(1,2)$ and that $f_{xx}(1,2) = 2$, $f_{yy}(1,2) = 0$ and $f_{xy}(1,2) = -1$. Which of the following statements is most likely to be true?

i. $f_y(1,2) > f_y(1,2.01)$.

ii. $f_y(1,2) < f_y(1,2.01)$.

iii. $f_y(1,2) = f_y(1,2.01)$.

iv. It is impossible to determine the relationship between $f_y(1,2)$ and $f_y(1,2.01)$ from the given information.

(2) $x-4y-2z=5$ $\quad\quad$ $x=a+bt,\ y=2-2t,\ z=2-t$

$\vec{n}=\hat{i}-4\hat{j}-2\hat{k}$ $\quad\quad$ $\vec{v}=b\hat{i}-2\hat{j}-\hat{k}$

$$\vec{n}\parallel\vec{v}$$

$$\vec{n}=\lambda\vec{v}$$

$$1=\lambda b,\quad -4=\lambda(-2),\quad -2=\lambda(-1)$$

$$\Rightarrow \lambda=2$$

$$1=2b$$

$$\boxed{1/2=b}$$

(3) $L(x,y)=f(10,20)+f_x(10,20)(x-10)+f_y(10,20)(y-20)$

$L(x,y)=1{,}000{,}000+2(10)(x-10)+(20)^2(y-20)$

$=1{,}000{,}000+20(x-10)+400(y-20)$

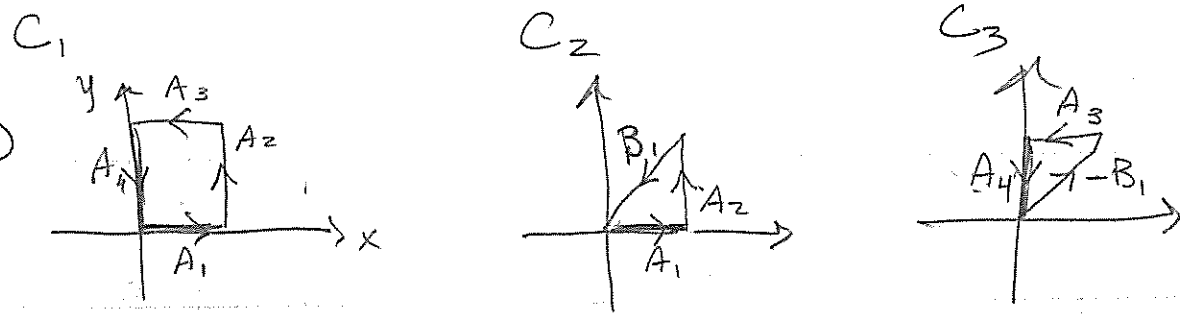
$f(12,19)\approx L(12,19)=1{,}000{,}000+20(2)+400(-1)$

$=\boxed{999{,}640}$

(4) When f is maximum subject to constraint g, we should have $\nabla f(2,5)=\lambda\nabla g(2,5)$. But there is no λ such that

$$4\hat{i}-10\hat{j}=\lambda(2\hat{i}+5\hat{j})$$

5



$$\oint_{C_1} \vec{F}\cdot d\vec{r} = \int_{A_1} \vec{F}\cdot d\vec{r} + \int_{A_2} \vec{F}\cdot d\vec{r} + \int_{A_3} \vec{F}\cdot d\vec{r} + \int_{A_4} \vec{F}\cdot d\vec{r}$$

(*) $$\oint_{C_2} \vec{F}\cdot d\vec{r} = \int_{A_1} \vec{F}\cdot d\vec{r} + \int_{A_2} \vec{F}\cdot d\vec{r} + \int_{B_1} \vec{F}\cdot d\vec{r}$$

(**) $$\oint_{C_3} \vec{F}\cdot d\vec{r} = \int_{-B_1} \vec{F}\cdot d\vec{r} + \int_{A_3} \vec{F}\cdot d\vec{r} + \int_{A_4} \vec{F}\cdot d\vec{r}$$

But $$\int_{-B_1} \vec{F}\cdot d\vec{r} = -\int_{B_1} \vec{F}\cdot dr$$

So when we add (*) and (**) the line integrals over B_1 & $-B_1$ cancel, leaving exactly the sum that $\oint_{C_1} \vec{F}\cdot d\vec{r}$ equals.

So $\boxed{\text{TRUE}}$

(6) $H_x = 2\cos(2x+y)$ $\quad$ $H_x(\pi,\pi,-5) = 2\cos(3\pi)$

$= -2$

$H_y = \cos(2x+y)$ $\quad$ $H_y(\pi,\pi,-5) = -1$

$H_z = 1$ $\quad$ $H_z(\pi,\pi,-5) = 1$

$$-2(x-\pi) - 1(y-\pi) + 1(z+5) = 0$$

(7) critical points when $\nabla f = \vec{0}$

(a) or when ∇f is undefined

$f_x = 5x - y + 15$ $\quad$ $f_y = -x + \frac{1}{25}y^2 - 3$

$5x - y + 15 = 0$ $\quad$ $-x + \frac{1}{25}y^2 - 3 = 0$

$y = 5(x+3)$ $\quad$ $-x + \frac{1}{25}(25)(x+3)^2 - 3 = 0$

$-x + x^2 + 6x + 9 - 3 = 0$

$x^2 + 5x + 6 = 0$

$(x+3)(x+2) = 0$

$x = -3, -2$

$x = -3 \Rightarrow y = 5(0) = 0$

$x = -2 \Rightarrow y = 5(1) = 5$

critical points are $(-3,0), (-2,5)$.

note: ∇f is always defined.

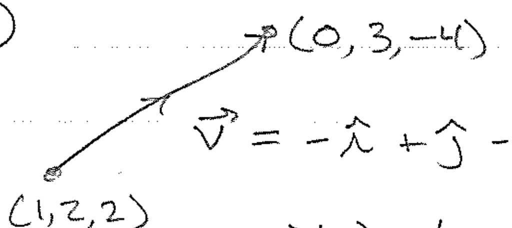
7b $f_{xx} = 5, \quad f_{yy} = \frac{2}{25} y \quad , f_{xy} = -1$

$D = 5\left(\frac{2}{25} y\right) - (-1)^2 = \frac{2}{5} y - 1$

$D(-3,0) = -1 < 0 \Rightarrow (-3,0)$ is a saddle pt.

$D(-2,5) = 1 > 0 \,,\ f_{xx} = 5 > 0 \Rightarrow (-2,5)$ is a local min.

8



$\vec{v} = -\hat{\imath} + \hat{\jmath} - 6\hat{k}$

$\vec{r}(t) = (1-t)\hat{\imath} + (2+t)\hat{\jmath} + (2-6t)\hat{k}, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = -\hat{\imath} + \hat{\jmath} - 6\hat{k}$

$\vec{F}(\vec{r}(t)) = (3-7t)\hat{\imath} + (6-18t)\hat{\jmath} + (12+6t)\hat{k}$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = -3 + 7t + 6 - 18t - 72 - 36t$

$= -69 - 47t$

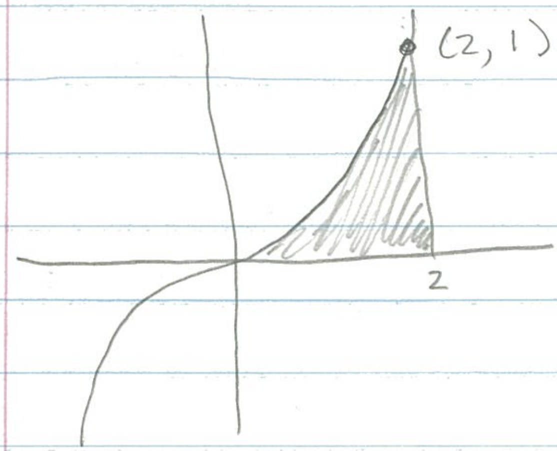
$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$= \int_0^1 (-69 - 47t)\, dt = -69t - \frac{47}{2} t^2 \Big]_0^1$

$= -69 - \frac{47}{2} = \boxed{-92.5}$

(9a) $\int_0^1 \int_{(8y)^{1/3}}^{2} \frac{1}{1+x^4}\,dx\,dy$

$x=(8y)^{1/3} \longrightarrow \frac{x^3}{8}=y$



$\int_0^2 \int_0^{\frac{x^3}{8}} \frac{1}{1+x^4}\,dy\,dx$

(b) $= \int_0^2 \left[\frac{y}{1+x^4} \right]_0^{x^3/8} dx = \int_0^2 \frac{1}{8} \left[\frac{x^3}{1+x^4} \right] dx$

$u = 1+x^4$

$du = 4x^3\,dx$

$\frac{1}{4}\,du = x^3\,dx$

$\frac{1}{8}\int \frac{1}{4} \frac{du}{u} = \frac{1}{32} \ln|u|$

$\frac{1}{32} \ln|1+x^4| \Big]_0^2 = \frac{1}{32}(\ln 17 - \ln 1) = \frac{\ln 17}{32} \approx .089$

(10) Arlington MA