Math 106
Final Exam
12/12/12

Name ____________________

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable. Give *exact* answers unless otherwise noted.

1. (12 pts.) Recall that the Taylor series for $\arctan x$ centered at $x_0 = 0$ is

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \text{ on } [-1, 1].$$

   (a) Find the first four non-zero terms of the Taylor series for $f(x) = x^3 \arctan(-x^2)$ centered at $x_0 = 0$.

   (b) Compute a partial sum for the series above to estimate $f(1)$ with error less than $\frac{1}{10}$.

2. (7 pts.) A cistern has the shape of the lower half of a sphere of radius 5 ft and is half full of water. *Set up, but do not evaluate*, the integral which represents the work required to pump all of the water to a point 4 feet above the top of the cistern. Recall: the weight of water is 62.4 lb/ft^3.

3. (6 pts.) Use separation of variables to solve the following initial value problem. Be sure to use the initial condition to determine the exact value of any constant you introduce.

$$\text{for } -1 < y < 1, \quad \frac{dy}{dt} = 2t\sqrt{1-y^2} \ \text{ with } \ y(0) = \frac{1}{2}$$

4. (7 pts. each) Evaluate the following integrals. Give exact values please.

(a) $\displaystyle\int_0^{\pi/3} x \sec^2 x \, dx$

(b) $\displaystyle\int_0^{12} \frac{dx}{\sqrt{25 + x^2}}$

5. (10 pts.) Consider $\displaystyle\sum_{k=1}^{\infty} \frac{8}{(4k-3)(4k+1)}$.

(a) Use partial fractions to rewrite $a_k = \dfrac{8}{(4k-3)(4k+1)}$.

(b) Find the first 3 terms for the sequence of ***partial sums*** for $\displaystyle\sum_{k=1}^{\infty} a_k$. Be sure to use your answer for a_k from part (a). Give exact values, no decimals—this will help you to identify the pattern and general formula required for part (c) below.

$S_1 =$

$S_2 =$

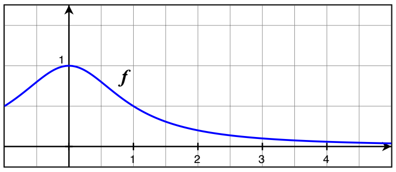
$S_3 =$

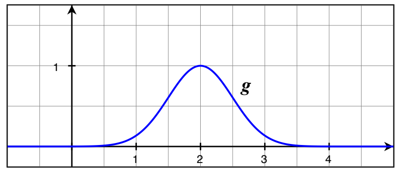
(c) Use part (b) to find a formula for the n^{th} partial sum, S_n. Then find $\displaystyle\lim_{n\to\infty} S_n$.

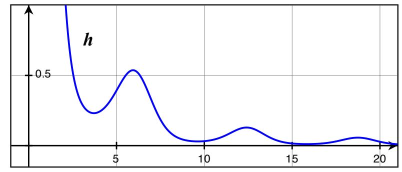
(d) What does this tell you about $\displaystyle\sum_{k=1}^{\infty} \frac{8}{(4k-3)(4k+1)}$?

6. (12 pts.) Consider the following functions and their graphs:

$$f(x) = \frac{1}{1+x^2} \qquad g(x) = e^{-2(x-2)^2} \qquad h(x) = \frac{e^{2+\cos x}}{x^2}$$







Each of the improper integrals below converge.

$$\int_1^\infty f(x)\,dx, \qquad \int_1^\infty g(x)\,dx, \qquad \text{and} \int_1^\infty h(x)\,dx$$

(a) Can you use the integral test to determine if $\displaystyle\sum_{k=1}^{\infty} \frac{1}{1+k^2}$ converges? Explain your answer.

(b) Can you use the integral test to determine if $\displaystyle\sum_{k=1}^{\infty} e^{-2(k-2)^2}$ converges? Explain your answer.

(c) Can you use the integral test to determine if $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2+\cos k}}{k^2}$ converges? Explain your answer.

7. (13 pts.) Find the interval of convergence for the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k2^k}(x+3)^k$. *Be sure to check endpoints.*

8. (7 pts.) The shelf life, x (in hours), of a certain perishable packaged food is a random variable whose probability density function is given by $p(x) = \frac{20,000}{(x+100)^3}$ for $x > 0$. Find the probability that one of these packages will have a shelf life of more than 200 hours.

9. (4 pts. each) Complete the following.

(a) Suppose f is increasing, but concave down. If a trapezoid sum is used to approximate $\int_a^b f(x)\,dx$, then the approximation will be an *overestimate* / *underestimate* (circle one) because ...

(b) Taylor polynomials are designed to have certain features in common with the function they are approximating. If $P_2(x)$ is the second order Taylor polynomial approximating $f(x)$ for x near a, then the graphs of $P_2(x)$ and $f(x)$ will have the following features in common...

(c) The series $\frac{6}{5}-\frac{18}{25}+\frac{54}{125}-\frac{162}{625}+\cdots$ may be written using summation notation as ____________________.

This series *converges* / *diverges* (circle one) because ...

10. (7 pts.) Consider the region bounded by $y + x = 2$ and $y + 2x^2 = 3$. Set up, *but do not evaluate*, the integral that represents the volume of the solid found by revolving this region about the line $y = 4$.

BONUS: DO ONE OF THE FOLLOWING:

- Refer to Problem #1 on page 1: What is $f^{(105)}(0)$? Justify your answer.
- Use a comparison to determine if $\displaystyle\sum_{k=1}^{\infty} \frac{e^{2+\cos k}}{k^2}$ converges.

Formulae you may find useful.

- $\displaystyle\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$

- $\displaystyle\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$

- $\displaystyle\int \cot\theta\, d\theta = -\ln|\csc\theta| + C$

- $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$

- Log Properties
  - $\ln(xy) = \ln x + \ln y$
  - $\ln \frac{x}{y} = \ln x - \ln y$
  - $\ln x^y = y \ln x$

- Trigonometric Identities
  - $\sin^2\theta + \cos^2\theta = 1$
  - $\sin^2\theta = \frac{1}{2}(1 - \cos(2\theta))$
  - $\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta))$
  - $\sec^2\theta = 1 + \tan^2\theta$
  - $\sin(2\theta) = 2\sin\theta\cos\theta$
  - $\cos(2\theta) = cos^2\theta - \sin^2\theta$