Fall

Math 106 ~~Winter~~ 2013

Final Exam (75 points)

Name: Solutions.

Show all your work to receive full credit for a problem and keep your written answers brief and clear. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

There are twelve questions on five pages. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$ $\qquad \int u\,dv = uv - \int v\,du$

$|I-L_n| \le \dfrac{K_1(b-a)^2}{2n}$ $\qquad |I-R_n| \le \dfrac{K_1(b-a)^2}{2n}$

$|I-T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$ $\qquad |I-M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$

$$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

$$|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}\,|x-x_0|^{n+1}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p \le 1$.

$\int_0^1 \frac{1}{x^p}\,dx$ converges for $p<1$ and diverges for $p \ge 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \le 1$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for x in $(-\infty,\infty)$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for x in $(-\infty,\infty)$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for x in $(-\infty,\infty)$.

1. (6 points) Solve the following initial value problem:

$$(\ln x)^2 \frac{dy}{dx} = \frac{1}{x}, \quad y(2) = 0.$$

$$dy = \frac{1}{x} \cdot \frac{1}{(\ln x)^2} dx$$

$$\int dy = \int \frac{1}{x} \cdot \frac{1}{(\ln x)^2} dx.$$

For $\int \frac{1}{x} \cdot \frac{1}{(\ln x)^2} dx$, let $u = \ln x$, $du = \frac{1}{x} dx$.

$$\int \frac{1}{x} \cdot \frac{1}{(\ln x)^2} dx = \int \frac{1}{u^2} \cdot du = \frac{u^{-1}}{-1} = \frac{-1}{\ln x} + C.$$

So $y = \frac{-1}{\ln x} + C.$

When $x = 2$, $y = 0$.

$0 = \frac{-1}{\ln 2} + C. \quad C = \frac{1}{\ln 2}.$

Hence $y = \frac{-1}{\ln x} + \frac{1}{\ln 2}$

2. (6 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{3x^2+x+23}{(4-x)(x^2+9)}\,dx$$

$$\frac{3x^2+x+23}{(4-x)(x^2+9)} = \frac{A}{4-x} + \frac{Bx+C}{x^2+9}$$

$$3x^2+x+23 = A(x^2+9) + (Bx+C)(4-x).$$

$\underline{x=4:}$ $48+4+23 = 25A+0.$ $\boxed{A=3}.$

$\underline{x=0:}$ $23 = 9A+4C.$ $23 = 27+4C.$ $\boxed{C=-1}.$

$\underline{x=1:}$ $3+1+23 = 10A+(B+C)3.$ $27 = 30+3B-3$ $\boxed{B=0}.$

$$\int \frac{3x^2+x+23}{(4-x)(x^2+9)}\,dx = \int \frac{3}{4-x}\,dx - \int \frac{1}{x^2+9}\,dx$$

$u = 4-x$

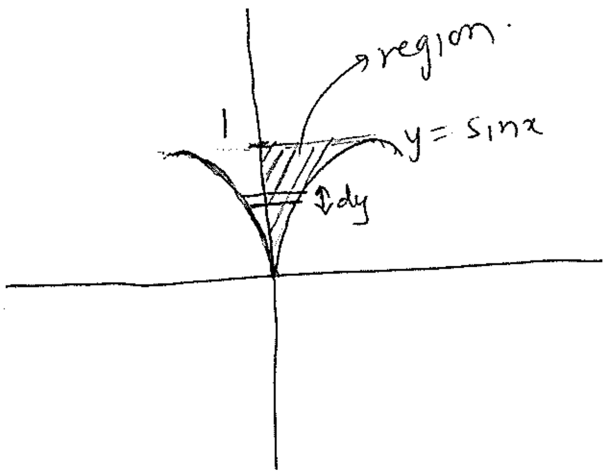
$du = -dx.$

$$= \int \frac{3}{u}(-du) - \frac{1}{3}\arctan\left(\frac{x}{3}\right)$$ formula 13, $a=3$.

$$= -3\ln|u| - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C$$

$$= -3\ln|4-x| - \frac{1}{3}\arctan\left(\frac{x}{3}\right) + C.$$

3. (4 points) Sketch the region bounded by the curve $y = \sin x$, the line $y = 1$ and the y-axis. Write (but do not evaluate) an integral to find the volume of the solid that is formed when this region is rotated about the y-axis.



Radius of slice $= x$

~~$= \text{arctan}$~~

$= \arcsin(y)$ (since $y = \sin x$).

Volume of slice $= \pi (\arcsin y)^2 \cdot dy$

Volume of solid

$$= \int_0^1 \pi (\arcsin y)^2 dy .$$

4. (4 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \tan^4 x \sec^4 x \, dx$$

$u = \tan x \qquad du = \sec^2 x \, dx$.

$\int \tan^4 x \sec^4 x \, dx = \int u^4 \sec^2 x \cdot du$

$= \int u^4 (\tan^2 x + 1) \, du$

$= \int u^4 (u^2 + 1) \, du$

$= \int (u^6 + u^4) \, du$

$= \frac{u^7}{7} + \frac{u^5}{5} = \frac{(\tan x)^7}{7} + \frac{(\tan x)^5}{5} + C$.

5. (6 points) Suppose a function f is such that $0 < f'(x) \leq 6$ for all x in $[-2, 3]$.

Let $I = \int_{-2}^{3} f(x)\, dx$. Use this information to answer the following questions.

(a) If possible, order the quantities I, T_{50}, M_{50} from least to greatest, where T_{50} is the trapezoid approximation of I with 50 subintervals and M_{50} is the midpoint approximation with 50 subintervals. If it is not possible to order them with the given information, explain why.

We have no information about the concavity of $f(x)$ and we need to know this ~~if~~ to determine whether T_{50} and M_{50} overestimate or underestimate the value of I.

So we cannot order I, T_{50}, M_{50}.

(b) What is the least value of n which guarantees that a right sum approximation R_n approximates I within ± 0.01? Justify your answer.

$|I - R_n| \leq \dfrac{K_1 (b-a)^2}{2n}$. We want $|I - R_n| \leq 0.01$.

So let's solve for n: $\dfrac{K_1 (b-a)^2}{2n} \leq 0.01$

$a = -2, b = 3$. $0 < f'(x) \leq 6$ for all x in $[-2, 3]$.

So $K_1 = 6$ (since $\max |f'(x)| = 6$ on $[-2,3]$)

$\dfrac{6(5)^2}{2n} \leq 0.01$.

$\dfrac{75}{n} \leq 0.01$.

$n \geq \dfrac{75}{0.01} = 7500$.

Hence least value of n for the desired approximation is 7500.

6. (5 points) The probability density function of a continuous random variable, X, is given by $f(x) = k(1 - \cos(2\pi x))$ for $0 \le x \le 100$ (the function is zero for all other values of x). Find k.

We know $\int_0^{100} f(x)\,dx = 1$.

$$\int_0^{100} k(1-\cos(2\pi x))\,dx$$

$$= k\int_0^{100} (1-\cos(2\pi x))\,dx.$$

~~$u = \cos 2\pi x$ $du = -\sin(2\pi x)\cdot(2\pi)\,dx$~~

~~$\int(1-\cos(2\pi x)$~~

Let $u = 2\pi x$, $du = 2\pi\,dx$.

$$\int(1-\cos(2\pi x))dx = \int(1-\cos u)\frac{du}{2\pi}$$

$$= \frac{1}{2\pi}[u - \sin u] = \frac{1}{2\pi}[2\pi x - \sin(2\pi x)]$$

$$= x - \frac{\sin(2\pi x)}{2\pi} + C.$$

$$k\int_0^{100}(1-\cos(2\pi x))\,dx$$

$$= k\left[x - \frac{\sin(2\pi x)}{2\pi}\right]_0^{100}$$

$$= k[100 - 0 - (0-0)] = 100k.$$

Hence $100k = 1$.

So $k = \frac{1}{100}$.

7. (7 points) Use comparisons to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{8n+3}.$$

$8n+3 \le 8n+3n$ for all $n \geq 1$.

So for $n \geq 1$, $\frac{1}{8n+3} \geq \frac{1}{11n} \geq 0$.

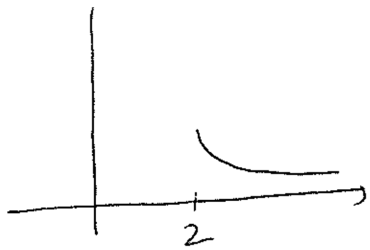
$$\frac{\sqrt{n}+1}{8n+3} \geq \frac{\sqrt{n}}{11n} \quad (\text{since } \sqrt{n}+1 \geq \sqrt{n}).$$

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{11n} = \sum_{n=1}^{\infty} \frac{1}{11\sqrt{n}} \text{ diverges } (\text{since } p = \tfrac{1}{2} < 1).$$

Hence by comparison test, $\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{8n+3}$ diverges.

8. (8 points) Determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{k=2}^{\infty} \frac{k}{e^{3k}}.$$



Graph of $\frac{x}{e^{3x}}$ for $x \geq 2$.

$\frac{x}{e^{3x}}$ is continuous, positive and decreasing on $[2, \infty)$. So integral test can be used for this problem.

To find $\int_2^{\infty} \frac{x}{e^{3x}}\,dx = \int_2^{\infty} x e^{-3x}\,dx$, first find $\int x e^{-3x}\,dx$.

Integration by parts: $u = x$, $dv = e^{-3x}\,dx$.
$du = dx$, $v = \int e^{-3x}\,dx = \frac{e^{-3x}}{-3}$

$$\int x e^{-3x}\,dx = x \cdot \frac{e^{-3x}}{-3} - \int \frac{e^{-3x}}{-3}\,dx = -\frac{x e^{-3x}}{3} + \frac{1}{3} \cdot \frac{e^{-3x}}{(-3)} + C$$

$$= \frac{-x e^{-3x}}{3} - \frac{e^{-3x}}{9} + C.$$

$$\int_2^t x e^{-3x}\,dx = \left[\frac{-x e^{-3x}}{3} - \frac{e^{-3x}}{9}\right]_2^t = \frac{-t e^{-3t}}{3} - \frac{e^{-3t}}{9} + \frac{2e^{-6}}{3} + \frac{e^{-6}}{9}.$$

As $t \to \infty$, $-te^{-3t} \to 0$, $-e^{-3t} \to 0$.

$\lim_{t\to\infty} -e^{-3t} = \lim_{t\to\infty} \frac{-1}{e^{3t}} = 0$. $\lim_{t\to\infty} -te^{-3t} = \lim_{t\to\infty} \frac{-t}{e^{-3t}} \overset{\text{L'Hôpital's rule}}{=} \lim_{t\to\infty} \frac{-1}{-3e^{-3t}} = 0$

So $\lim_{t\to\infty} \int_2^t x e^{-3x}\,dx = 0 + \frac{2}{3}e^{-6} + \frac{1}{9}e^{-6} = \frac{7}{9}e^{-6}$. Hence integral converges.

So by integral test, $\sum_{k=2}^{\infty} \frac{k}{e^{3k}}$ converges.

Upper bound $= \frac{2}{e^6} + \frac{7}{9}e^{-6} = \left(2 + \frac{7}{9}\right)e^{-6} = \frac{25}{9}e^{-6}$.

first term of series.

9. (6 points) Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n n!}{\ln n}$ converge absolutely or conditionally? Explain.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!}{\ln(n+1)} \cdot \frac{\ln n}{n!} = \lim_{n\to\infty} (n+1) \cdot \frac{\ln n}{\ln(n+1)}.$$

$$\lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} \overset{\text{L'Hôpital's rule}}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+1}} = \lim_{n\to\infty} \frac{n+1}{n} \overset{\text{L'Hôpital's rule}}{=} \lim_{n\to\infty} \frac{1}{1} = 1.$$

So $\lim_{n\to\infty} (n+1) \cdot \frac{\ln n}{\ln(n+1)} = \infty > 1$. Hence $\sum_{n=2}^{\infty} \frac{(-1)^n n!}{\ln n}$ diverges.

So it does not converge absolutely and does not converge conditionally.

10. (6 points) The sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$ is given by $S_n = 21\left(\left(\frac{2}{3}\right)^n - 1\right)$ for $n = 1, 2, \ldots$. Use this information to answer the questions that follow.

(a) Does the sequence S_n converge? Explain.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 21\left(\left(\frac{2}{3}\right)^n - 1\right) = 21(0-1) = -21.$$

As $n\to\infty$, $\left(\frac{2}{3}\right)^n \to 0$.

So the sequence S_n converges.

(b) Does the series $\sum_{n=0}^{\infty} a_n$ converge? Explain.

Since the sequence S_n converges, ~~the~~ $\sum_{n=0}^{\infty} a_n$ converges. (by definition of convergence of series.)

(c) If possible, find $\lim_{n\to\infty} a_n$. If it is not possible to do so, explain why.

Since $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$.

(because if $\lim_{n\to\infty} a_n \neq 0$, then series must diverge which is not the case here.)

11. (9 points) Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^3}$.

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(x-5)^{k+1}}{(k+1)^3}\right| \cdot \left|\frac{k^3}{(x-5)^k}\right|$$

$$= \lim_{k\to\infty} \frac{k^3}{(k+1)^3} \cdot |x-5| = |x-5|.$$

~~$= \lim_{k\to\infty}$~~ ~~k~~

As $k\to\infty$, $\frac{k^3}{(k+1)^3} \longrightarrow 1$.

Look at graph. and show it.
(or use algebraic method.)

So series converges if $|x-5|<1$ and diverges if $|x-5|>1$.

$|x-5|<1$ So $-1<x-5<1$. So $4<x<6$.

Ratio test is inconclusive if $|x-5|=1$. ie if $x=4$ or $x=6$.

So check convergence at these points separately

$\underline{x=4:}$ $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^3} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$.

Alternating series. So use AST. $c_k = \left|\frac{(-1)^k}{k^3}\right| = \frac{1}{k^3}$.

$\lim_{k\to\infty} c_k = \lim_{k\to\infty} \frac{1}{k^3} = 0$. $c_{k+1} = \frac{1}{(k+1)^3} \leq \frac{1}{k^3}$ (Since $(k+1)^3 \geq k^3$)

So $c_1 \geq c_2 \geq c_3 \geq \cdots$

Hence by AST, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3}$ converges.

$\underline{x=6:}$ $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^3} = \sum_{k=1}^{\infty} \frac{1}{k^3}$ converges ($p=3>1$).

So interval of convergence is $[4,6]$.

12. (8 points) Let $f(x) = x^2 e^{(x^2)}$. Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for f. Also write this series using sigma notation.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

$$e^{(x^2)} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(x^2)^k}{k!}$$

$$\text{So } e^{(x^2)} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k}}{k!}$$

$$x^2 e^{(x^2)} = x^2 + x^4 + \frac{x^6}{2!} + \frac{x^8}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{x^{2k+2}}{k!}$$

(b) Use the series in part (a) to write the first four non-zero terms of the series for $\int_0^2 x^2 e^{(x^2)}\, dx$. Also write this series using sigma notation.

$$\int x^2 e^{(x^2)} dx = \frac{x^3}{3} + \frac{x^5}{5} + \frac{1}{2!}\frac{x^7}{7} + \frac{1}{3!}\frac{x^9}{9} + \cdots = \sum_{k=0}^{\infty} \frac{1}{k!}\frac{x^{2k+3}}{(2k+3)}$$

$$\text{So } \int_0^2 x^2 e^{(x^2)} dx = \frac{2^3}{3} + \frac{2^5}{5} + \frac{1}{2!}\frac{2^7}{7} + \frac{1}{3!}\frac{2^9}{9} + \cdots - (0 + 0 + 0 + 0 + \cdots)$$

$$= \frac{2^3}{3} + \frac{2^5}{5} + \frac{1}{2!}\frac{2^7}{7} + \frac{1}{3!}\frac{2^9}{9} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\frac{2^{2k+3}}{(2k+3)}$$