Math 106 Fall 2013

Final Exam (75 points)

Name: ______________________________

Show all your work to receive full credit for a problem and keep your written answers brief and clear. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

There are twelve questions on five pages. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$\int u\,dv = uv - \int v\,du$

$|I-L_n| \le \dfrac{K_1(b-a)^2}{2n}$

$|I-R_n| \le \dfrac{K_1(b-a)^2}{2n}$

$|I-T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$

$|I-M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$

$$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

$$|f(x)-P_n(x)| \le \frac{K_{n+1}}{(n+1)!}\,|x-x_0|^{n+1}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p\le 1$.

$\int_0^1 \frac{1}{x^p}\,dx$ converges for $p<1$ and diverges for $p\ge 1$.

$\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p>1$ and diverges for $p\le 1$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for x in $(-\infty,\infty)$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for x in $(-\infty,\infty)$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for x in $(-\infty,\infty)$.

1. (6 points) Solve the following initial value problem:

$$(\ln x)^2 \frac{dy}{dx} = \frac{1}{x}\ , \ \ y(2) = 0.$$

2. (6 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{3x^2 + x + 23}{(4-x)(x^2+9)}\, dx$$

3. (4 points) Sketch the region bounded by the curve $y = \sin x$, the line $y = 1$ and the y-axis. Write (but do not evaluate) an integral to find the volume of the solid that is formed when this region is rotated about the y-axis.

4. (4 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \tan^4 x \sec^4 x \, dx$$

5. (6 points) Suppose a function f is such that $0 < f'(x) \leq 6$ for all x in $[-2, 3]$.

Let $I = \int_{-2}^{3} f(x)\, dx$. Use this information to answer the following questions.

(a) If possible, order the quantities I, T_{50}, M_{50} from least to greatest, where T_{50} is the trapezoid approximation of I with 50 subintervals and M_{50} is the midpoint approximation with 50 subintervals. If it is not possible to order them with the given information, explain why.

(b) What is the least value of n which guarantees that a right sum approximation R_n approximates I within ± 0.01? Justify your answer.

6. (5 points) The probability density function of a continuous random variable, X, is given by $f(x) = k(1 - \cos(2\pi x))$ for $0 \leq x \leq 100$ (the function is zero for all other values of x). Find k.

7. (7 points) Use comparisons to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+1}{8n+3} \ .$$

8. (8 points) Determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{k=2}^{\infty} \frac{k}{e^{3k}} \ .$$

9. (6 points) Does the series $\sum_{n=2}^{\infty} \frac{(-1)^n n!}{\ln n}$ converge absolutely or conditionally? Explain.

10. (6 points) The sequence of partial sums of the series $\sum_{n=0}^{\infty} a_n$ is given by $S_n = 21\left(\left(\frac{2}{3}\right)^n - 1\right)$ for $n = 1, 2, \ldots$. Use this information to answer the questions that follow.

(a) Does the sequence S_n converge? Explain.

(b) Does the series $\sum_{n=0}^{\infty} a_n$ converge? Explain.

(c) If possible, find $\lim_{n\to\infty} a_n$. If it is not possible to do so, explain why.

11. (9 points) Find the interval of convergence of the series $\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^3}$.

12. (8 points) Let $f(x) = x^2 e^{(x^2)}$. Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for f. Also write this series using sigma notation.

(b) Use the series in part (a) to write the first four non-zero terms of the series for $\int_0^2 x^2 e^{(x^2)}\, dx$. Also write this series using sigma notation.