Name: Solutions

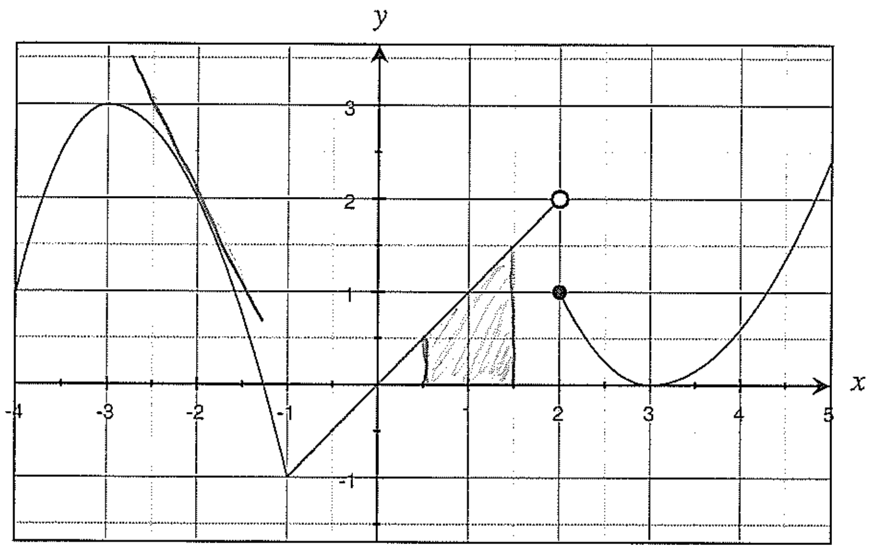
Math 105: Fall 2012
Final Exam

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable.

Good Luck!

Formulas for Common Geometric Shapes
- Circle: $A = \pi r^2$, $C = 2\pi r$ • Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$ • Cylinder: $V = \pi r^2 h$, lateral area $= 2\pi r h$

1. (12 points) The following is a graph of $f(x)$ on the interval $[-4, 5]$.



(a) Find $\lim_{x \to 2^-} f(x)$. 2

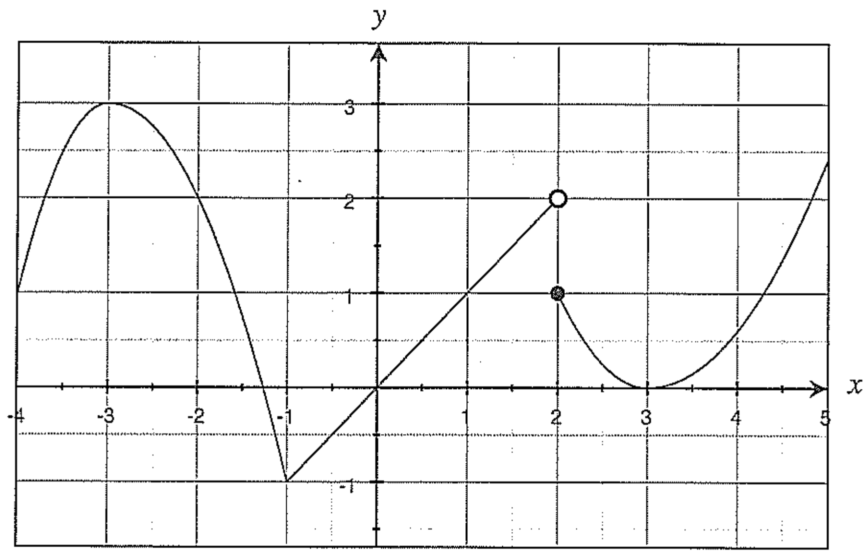
(b) Compute $\lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$. $= f'(1) =$ slope of line $= \boxed{1}$

(c) Use the graph to visually estimate $f'(-2)$. draw tangent line
estimate its slope $\boxed{-2}$

(d) Let $A(x) = \int_{1/2}^{x} f(t)\,dt$. Compute $A(1.5)$.

$A(1.5) = \int_{1/2}^{1.5} f(t)\,dt =$ shaded area $= \frac{1}{2}(1)(1/2 + 1.5) = \boxed{1}$

2. (12 points) The following is a graph of $f(x)$ on the interval $[-4, 5]$.



(a) Let F be an antiderivative of f.

i. Find the critical points of F. when $F' = f = 0$ or when $F' = f$ is undefined.

$f = 0$ at $x = -1.25, 0, 3$

f is never undefined

$\Rightarrow \boxed{x = -1.25, 0, 3}$

ii. Find the inflection points of F.

possible inflection points when $F'' = f' = 0$ or when $F'' = f'$ is undefined

$f' = 0$ at $x = -3, 3$

f' undefined at $x = -1, 2$

These were determined by looking at slope of f.

$F'' = f'$: $\oplus$ -3 $\ominus$ -1 $\oplus$ 2 $\ominus$ 3 $\oplus$

F: $\cup$ $\cap$ $\cup$ $\cap$ $\cup$

since concavity changes at all these points, they are all inflection points of F

$\boxed{x = -3, -1, 2, 3}$

(b) On what interval(s) is $f''(x)$ negative?

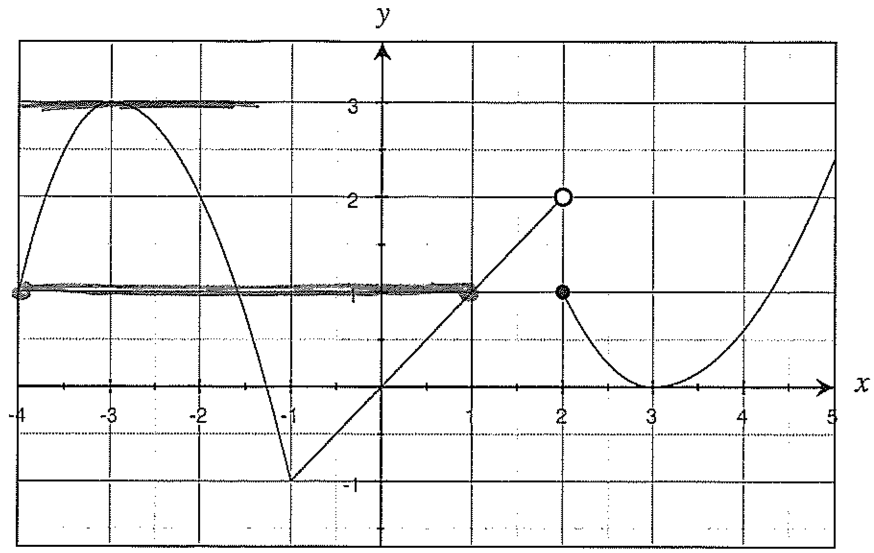
$f''(x)$ is negative when f is concave down

$\boxed{(-4, -1)}$

(c) On what interval(s) is $f'(x)$ decreasing?

$f'(x)$ is decreasing when $f''(x)$ is negative which by (b) is $\boxed{(-4, -1)}$

3. (8 points) The following is a graph of $f(x)$ on the interval $[-4, 5]$.



(a) Do the hypotheses of the Mean Value Theorem hold for f on the interval $[-4, 1]$? Explain.

One of the hypotheses of the MVT is that f be differentiable on the interval.
But f is not differentiable at $x=-1$, which is in the interval.

(b) Does the conclusion of the Mean Value Theorem hold for f on the interval $[-4, 1]$? Explain.

The conclusion is that there is some c between -4 and 1 such that $f'(c) = \dfrac{f(1)-f(-4)}{1--4}$.

$\dfrac{f(1)-f(-4)}{1--4} = 0 =$ slope of line between $(-4, f(-4))$ and $(1, f(1))$

Note that at $x=-3$ $f'(-3)=0$.

So yes the conclusion holds.

4. (20 points) Find the derivatives for each of the following.

(a) $f(x) = \sqrt[3]{x + \sqrt{x^e + e^\pi}}$. [You do NOT need to simplify your answer.]

$f(x) = (x + (x^e + e^\pi)^{1/2})^{1/3}$

$f'(x) = \frac{1}{3}(x + (x^e + e^\pi)^{1/2})^{-2/3}\left[1 + 1/2(x^e + e^\pi)^{-1/2}(ex^{e-1})\right]$

(b) $z(t) = \frac{2^{\ln t}}{e^{-t^2}}$ [You do NOT need to simplify your answer.]

$z'(t) = \frac{(\ln 2) 2^{\ln t}\left(\frac{1}{t}\right) e^{-t^2} - 2^{\ln t} e^{-t^2}(-2t)}{(e^{-t^2})^2}$

(c) $g(x) = \int_3^x 7 \arcsin t \, dt$ [You do NOT need to simplify your answer.]

$g'(x) = 7 \arcsin x$

(d) $y = x^{\sin x}$. [You do NOT need to simplify your answer.]

$\ln y = \sin x \ln x$

$\frac{1}{y}\frac{dy}{dx} = \sin x \frac{1}{x} + \cos x \ln x$

$\frac{dy}{dx} = x^{\sin x}\left(\sin x \frac{1}{x} + \cos x \ln x\right)$

5. (5 points) Find $\frac{dy}{dx}$ if $xy + 5 = \frac{x}{2} - 3\arctan y$.

$$x\frac{dy}{dx} + y = \frac{1}{2} - 3\frac{1}{1+y^2}\frac{dy}{dx}$$

$$x\frac{dy}{dx} + \frac{3}{1+y^2}\frac{dy}{dx} = \frac{1}{2} - y$$

$$\left(x + \frac{3}{1+y^2}\right)\frac{dy}{dx} = \frac{1}{2} - y$$

$$\frac{dy}{dx} = \frac{\frac{1}{2} - y}{x + \frac{3}{1+y^2}}$$

6. (5 points) Compute $\int_0^\pi (\sin x + e^x + x^2)\,dx$.

$$= -\cos x + e^x + \frac{x^3}{3}\Big]_0^\pi$$

$$= -\cos\pi + e^\pi + \frac{\pi^3}{3} - \left[-\cos 0 + e^0 + 0\right]$$

$$= 1 + e^\pi + \frac{\pi^3}{3} - (-1 + 1)$$

$$= 1 + e^\pi + \frac{\pi^3}{3}$$

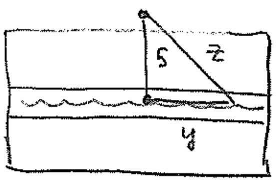
$$\approx 34.48$$

7. (9 points) Choose ONE of the following. If you do more than one, then clearly mark which one you want graded. If you don't, the WORST will be chosen for you.

(a) You are standing at the edge of a swimming pool and watching a friend who is swimming in a swim meet. Your friend is swimming in a straight line in a lane 5 feet from you. Your friend passes you and continues to swim at a speed of 4 feet per second. At the moment when your friend is 13 feet from you, at what rate is the distance between you and your friend changing? Be sure to express your answer using appropriate units.

(b) A stone dropped into a still pond sends out a circular ripple. The radius is increasing at a rate of 3 ft/sec. After 10 seconds, what is the rate of change of the area? Be sure to express your answer using appropriate units.

(a)



$\frac{dy}{dt} = 4$

$\frac{dz}{dt} = ?$

$5^2 + y^2 = z^2$

$2y\frac{dy}{dt} = 2z\frac{dz}{dt}$

$5^2 + y^2 = 13^2$

$y^2 = 169 - 25$

$y^2 = 144$

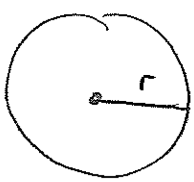
$y = 12$

$2(12)(4) = 2(13)\frac{dz}{dt}$

$\frac{48}{13} = \frac{dz}{dt}$

$\boxed{\frac{48}{13} \text{ ft/sec}} = 3.69 \text{ ft/sec}$

(b)



$\frac{dr}{dt} = 3$

$\frac{dA}{dt} = ?$

$A = \pi r^2$

$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(30)(3) = \boxed{180\pi \text{ ft}^2/\text{sec}}$

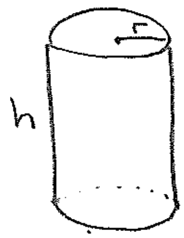
$= 565.5 \text{ ft}^2/\text{sec}$

10 secs

rate × time = distance

$3 \times 10 = 30$

8. (9 points) Use calculus to find the maximum possible volume of a cylinder if its total surface area (including both circular ends) is 150π. Be sure to confirm, using calculus, whether you have really found the maximum.



$V = \pi r^2 h$

$S.A. = 2\pi r h + 2\pi r^2 = 150\pi$

$rh + r^2 = 75$

$rh = 75 - r^2$

$h = \frac{75}{r} - r$

$V = \pi r^2 \left(\frac{75}{r} - r\right)$

$V = 75\pi r - \pi r^3$

$V' = 75\pi - 3\pi r^2 = 0$

$75\pi = 3\pi r^2$

$25 = r^2$

$5 = r \longrightarrow h = \frac{75}{5} - 5 = 15 - 5 = 10$

$V = \pi (5)^2 10 = \boxed{250\pi}$

$V'' = -6\pi r$

1 $V''(5) = -30\pi$ so V is concave down at $r = 5$ hence we have a max as desired.

9. (20 points) Let $f(x) = x^2 - 7$.

(a) Estimate $\int_2^5 f(x)\,dx$ using L_3, i.e., 3 rectangles and left-hand sums.

$\Delta x = \frac{5-2}{3} = 1$ $\quad L_3 = f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1$

$= (4-7) + (9-7) + (16-7)$

$= -3 + 2 + 9 = \boxed{8}$

(b) Determine $\int_2^5 f(x)\,dx$ using the FTC.

$\int_2^5 (x^2-7)\,dx = \frac{x^3}{3} - 7x\Big]_2^5 = \left[\frac{125}{3} - 35\right] - \left[\frac{8}{3} - 14\right]$

$= \frac{117}{3} - 21 = \boxed{18}$

(c) Use the limit definition of the definite integral to compute $\int_2^5 f(x)\,dx$, where $f(x) = x^2 - 7$.

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$R_n = \sum_{i=1}^{n} f\left(2 + \frac{3}{n}i\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n} \left[\left(2 + \frac{3}{n}i\right)^2 - 7\right]\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} \left[4 + \frac{12}{n} i + \frac{9}{n^2} i^2 - 7\right]\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n} \left[-\frac{9}{n} + \frac{36}{n^2} i + \frac{27}{n^3} i^2\right] = -\frac{9}{n}\sum_{i=1}^{n} 1 + \frac{36}{n^2}\sum_{i=1}^{n} i + \frac{27}{n^3}\sum i^2$$

$$= -\frac{9}{n}(n) + \frac{36}{n^{\cancel{2}}}\,\frac{\cancel{n}(n+1)}{2} + \frac{\overset{9}{\cancel{27}}}{n^{\cancel{3}2}}\,\frac{\cancel{n}(n+1)(2n+1)}{\cancel{6}2}$$

$$= -9 + \frac{18n + 18}{n} + \frac{9}{2}\,\frac{2n^2 + 3n + 1}{n^2}$$

$$= -9 + 18 + \frac{18}{n} + 9 + \frac{27}{2n} + \frac{9}{2n^2}$$

$$\int_2^5 f(x)dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(18 + \frac{18}{n} + \frac{27}{2n} + \frac{9}{2n^2}\right) = \boxed{18}$$