FINAL

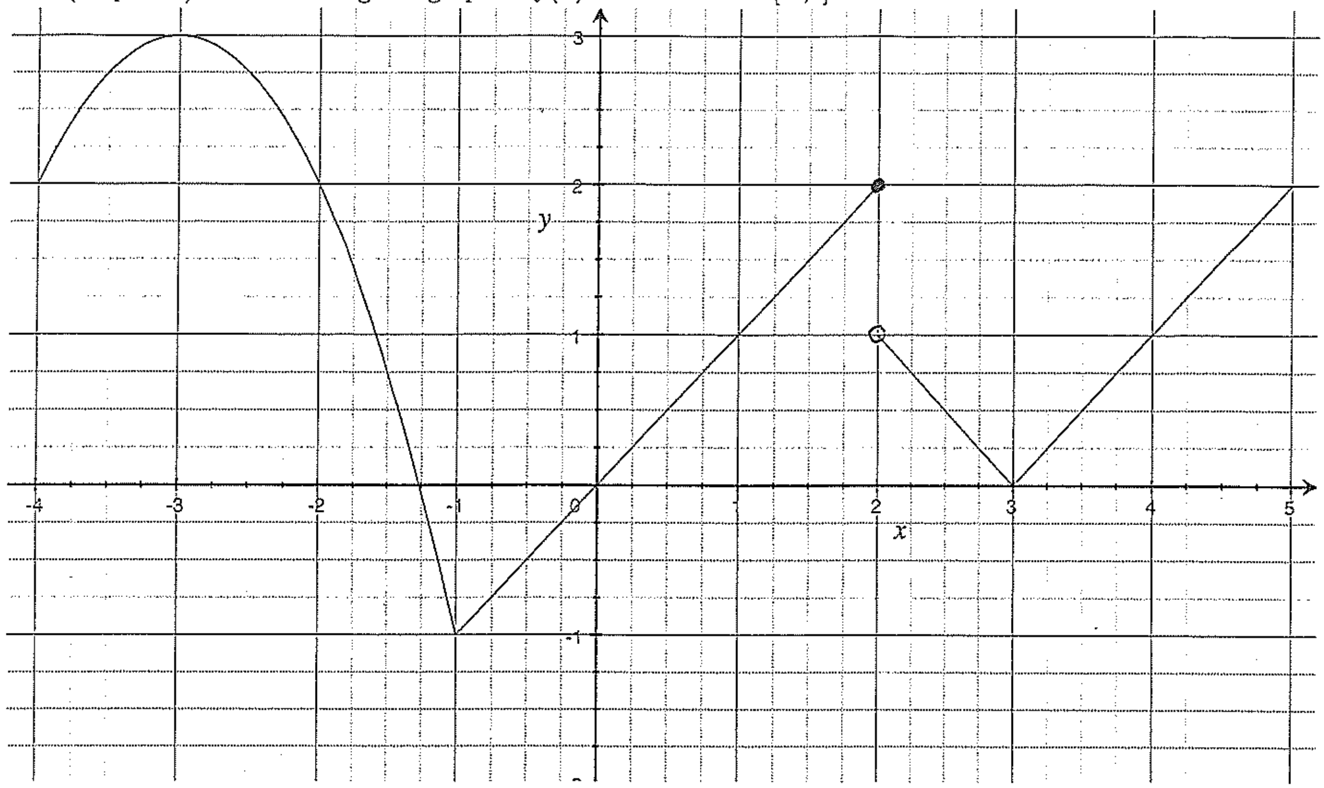
Math 105
12/11/12

Name: $\underbrace{\text{Key}}_{\text{by writing my name I swear this work is my own}}$

Read all of the following information before starting the exam:

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 10 problems and is worth 100 points, It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. (*15 points*) The following is a graph of $f(x)$ on the interval [-4,5].



a. (*2 pts*) Where is $f'(x)$ positive?

$(-4,-3) \cup (-1,2) \cup (3,5)$

b. (*2 pts*) Where is $f'(x)$ decreasing?

$(-4,-1)$

c. (*2 pts*) Where is $f''(x)$ negative?

$(-4,-1)$

d. (*2 pts*) Where is $f'(x)$ negative?

$(-3,-1) \cup (2,3)$

e. (*2 pts*) Does the hypothesis of the Mean Value Theorem hold on the interval [-4,2]? Explain.

No, the function is not differentiable on $(-4,2)$.

f. (*2 pts*) Does the conclusion of the Mean Value Theorem hold on the interval [-4,2]? Explain.

$\frac{f(b)-f(a)}{b-a} \rightarrow \frac{f(-4)-f(2)}{6} = \frac{0}{6} = 0$, just let $c=-3$ then $f'(c)=0$ ✓ yes!

g. (*3 pts*) Does the conclusion of the Intermediate Value Theorem hold on the interval [-4,3]? Explain.

$f(-4)=2$
$f(3)=0$

yes, every value between 0 and 2 is attained by an x-value between -4 & 3.

2. *(10 points)*

x	$f(x)$	$g(x)$	$j(x)$	$f'(x)$	$g'(x)$	$j'(x)$
-2	-1	1	-1	3	2	1
-1	1	3	2	-1	3	-2
0	2	1	1	-2	-2	2
1	3	1	-1	-1	3	1
2	-2	2	1	3	2	3
3	-1	1	-1	1	-2	2

a. *(5 pts)* $H(x) = \sqrt{f(x^2)} + ln(j(x))$. Find $H'(1)$.

$$H'(x) = \frac{1}{2}\left(f(x^2)\right)^{-1/2} \cdot f'(x^2) \cdot 2x + \frac{1}{j(x)} \cdot j'(x)$$

$$H'(1) = \frac{1}{2}(3)^{-1/2} \cdot (-1)(2) + \frac{1}{-1} \cdot 1$$

$$= \frac{-1}{\sqrt{3}} - 1 \qquad \boxed{\approx -1.577}$$

b. *(5 pts)* $F(x) = \dfrac{e^x g(x)}{f(x)^3}$. Find $F'(2)$.

$$F'(x) = \frac{\left(e^x g(x) + e^x g'(x)\right) f(x)^3 - \left[3f(x)^2 \cdot f'(x)\right] e^x g(x)}{f(x)^6}$$

$$F'(2) = \frac{(e^2 \cdot 2 + e^2 \cdot 2)(-8) - 3(4)(3)e^2 \cdot 2}{64}$$

$$= \frac{-32e^2 - 72e^2}{64} = \frac{-104e^2}{64} \approx \boxed{-12.007}$$

3. *(8 points)* Find the derivative of the following functions.

a. *(4 pts)* $g(s) = \sqrt[5]{s^3} + \frac{5}{s} + 2^{cos(s)} + \arctan(e^{3s} + 5\pi) + \ln(3)$

$$g'(s) = \frac{3}{5} s^{-2/5} - \frac{5}{s^2} + 2^{\cos(x)} \ln(2) \cdot (-\sin(x)) + \frac{1}{1 + (e^{3s} + 5\pi)^2} \cdot 3e^{3s}$$

b. *(4 pts)* $y = \dfrac{(2x+4)^3(x^2-2)^3 e^{-3x}}{(x-4)^4(5x^3-1)^2}$ (Use logarithmic differentiation)

$$\ln(y) = 3\ln(2x+4) + 3\ln(x^2-2) - 3x\ln(e) - 4\ln(x-4) - 2\ln(5x^3-1)$$

$$y = \left(\frac{6}{2x+4} + \frac{6x}{x^2-2} - 3 - \frac{4}{x-4} - \frac{30x^2}{5x^3-1}\right) y$$

4. (*7 points*) For the equation $x^3 + y^3 = ln(xy) - 1$ use implicit differentiation to find $\frac{dy}{dx}$.

$$3x^2 + 3y^2 y' = \frac{1}{xy}(y + xy')$$

$$3x^3 y + 3xy^3 y' = y + xy'$$

$$\frac{dy}{dx} = y' = \frac{y - 3x^3 y}{3xy^3 - x}$$

also

$$\frac{\frac{1}{x} - 3x^2}{3y^2 - \frac{1}{y}}$$

5. (*8 points*) Solve the following. Only use L'Hôpital's rule when appropriate. Show your work!!

a. (*4 pts*) $\lim_{x\to 0} \frac{\sqrt{4-x^2}-2}{x} \cdot \frac{\sqrt{4-x^2}+2}{\sqrt{4-x^2}+2}$

$$= \lim_{x\to 0} \frac{4 - x^2 - 4}{x(\sqrt{4-x^2}+2)} = \lim_{x\to 0} \frac{-x}{\sqrt{4-x^2}+2} = \frac{0}{4} = 0$$

b. (*4 pts*) $\lim_{x\to 0} x^x$

$$y = \lim_{x\to 0} x^x$$

$$\ln(y) = \lim_{x\to 0} x\ln(x) \quad 0 \cdot -\infty$$

$$= \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}} \stackrel{\frac{\infty}{\infty}}{=} \lim_{x\to 0} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0} -\frac{x^2}{x} = \lim_{x\to 0} -x = 0$$

$$\ln(y) = 0 \Rightarrow \boxed{y = 1}$$

6. (*7 points*) Find differentiable functions $f(x)$ and $g(x)$ such that $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$ and

a. (*3 pts*) $\lim_{x\to 5} \frac{f(x)}{g(x)} = 10$

$f(x) = 10x - 50$ $\quad \lim_{x\to 5} f(x) = 0$ $\quad \lim_{x\to 5} \frac{10x-50}{x-5} = 10$

$g(x) = x - 5$ $\quad \lim_{x\to 5} g(x) = 0$

b. (*4 pts*) $\lim_{x\to 5} \frac{f(x)}{g(x)} = \infty$

$f(x) = x - 5$ $\quad \lim_{x\to 5} f(x) = 0$ $\quad \lim_{x\to 5} \frac{x-5}{(x-5)^2} = \lim_{x\to 5} \frac{1}{x-5} \to \infty$

$g(x) = (x-5)^2$ $\quad \lim_{x\to 5} g(x) = 0$

7. (*8 points*) All edges of a cube are expanding at the same rate. The surface area is changing at a rate of 12 cm^2/second when each edge measures 3cm. Determine how fast the volume of the cube is changes when the edges are 3cm. (This is a two step problem)

$SA = 6x^2$

$\frac{dSA}{dt} = 12x\frac{dx}{dt} \Rightarrow 12 = 12(3)\frac{dx}{dt}$

$\Rightarrow$ at $x=3$, $\frac{dx}{dt} = 1/3 \text{ cm/sec}$

$V = x^3$

$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \frac{dV}{dt} = 3(3)^2\left(\frac{1}{3}\right) = 9$

$9 \text{ cm}^3/\text{sec}$

8. (*10 points*)

a. (*4 pts*) Determine the antiderivative of $2t + \frac{2t}{1+4t^4}$. Show your check.

$\boxed{F(t) = t^2 + \frac{1}{2}\arctan(2t^2) + C}$

check: $F'(t) = 2t + \frac{1}{2}\frac{1}{1+(4t^4)} \cdot 4t$ ✓

b. (*2 pts*) Find the derivative of $\ln(32 - 5x^2)$.

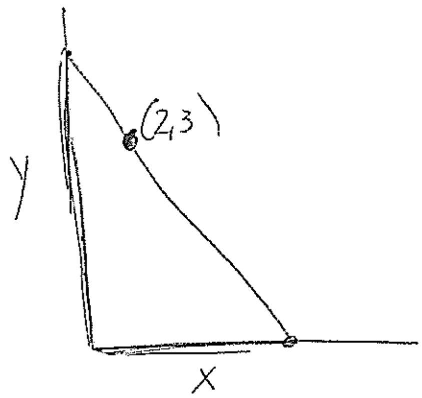
$(\ln(32-5x^2))' = \frac{1}{32-5x^2} \cdot -10x$

c. (*4 pts*) Use the FTC to determine $\int_0^2 \frac{x}{32-5x^2}dx$.

$F(x) = \frac{\ln|32-5x^2|}{-10}$

$\int_0^2 \frac{x}{32-5x^2}dx = \frac{\ln|32-5x^2|}{-10}\Big|_0^2 = \frac{\ln(12)}{-10} - \frac{\ln(32)}{-10} \approx .098$

9. (*10 points*) A right triangle is formed in the first quadrant by the $x-$ and y-axes and a line through the point (2,3). Find the vertices of the triangle so that its area is minimum. (Hint: The area of the triangle should be a function of the slope of the line. You don't know the slope.)



$A = \frac{1}{2} x y$

$y - 3 = m(x-2)$

The length x is the x-intercept. So when $y=0$,

$-3 = m(x-2)$

So, $x = \frac{-3}{m} + 2$

The length y is the y-intercept. So when $x=0$

$y - 3 = m(0-2)$

$y = -2m + 3$

$A = \frac{1}{2}\left(\frac{-3}{m}+2\right)(-2m+3)$

$= \frac{1}{2}\left(6 - \frac{9}{m} - 4m + 6\right)$

$= 6 - \frac{9}{2m} - 2m$

$A' = \frac{9}{2m^2} - 2 = 0 \Rightarrow m^2 = \frac{9}{4} \quad m = \pm\frac{3}{2}$

$A'' = \frac{-18}{2m^3}$ $< 0 \cap m = 3/2$

$> 0 \cup m = -3/2 \to \underline{\underline{\text{min}}}$ $\quad m = -3/2$

Vertices are $(0,6)$, $(0,0)$, and $(4,0)$.

10. (*17 points*) Let $f(x) = 3x^2 + 1$.

a. (*4 pts*) Estimate the area $\int_1^4 f(x)dx$ using 3 rectangles and left-hand sums.

$$L_3 = 1 \cdot (f(1)+f(2)+f(3)) = (4+13+28) = 45$$

b. (*3 pts*) Determine $\int_1^4 f(x)dx$ using the FTC.

$$= x^3 + x \Big|_1^4 = (4^3+4) - (1^3+1) = 66$$

c. (*10 pts*) Use infinite Riemann sums $\left(\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n f(x_i^*)\Delta x\right)$ to find $\int_1^4 f(x)dx$.

$$\sum_{i=1}^n 1 = n, \sum_{i=1}^n i = \frac{n(n+1)}{2}, \sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}, \sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$\Delta x = \frac{3}{n} \qquad x_i^* = 1 + \frac{3i}{n}$$

$$\int_1^4 f(x)dx = \lim_{n\to\infty} \sum_{i=1}^n \left[3\left(1+\frac{3i}{n}\right)^2 + 1\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^n \left[3\left(1 + \frac{6i}{n} + \frac{9i^2}{n^2}\right) + 1\right]\frac{3}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^n \frac{12}{n} + \sum_{i=1}^n \frac{54}{n^2} i + \sum_{i=1}^n \frac{81}{n^3} i^2$$

$$= \lim_{n\to\infty} \frac{12}{n}\sum_{i=1}^n 1 + \frac{54}{n^2}\sum_{i=1}^n i + \frac{81}{n^3}\sum_{i=1}^n i^2$$

$$= 12 + \frac{54}{2} + \frac{162}{6} = \boxed{66}$$