1. Determine if the series is convergent or divergent by making a comparison (DCT or LCT) with a suitable b_n. Fill in the blanks with your answer. For "Convergent or Divergent" write "Convergent" or "C" if the series to the left is convergent. Otherwise write "Divergent" or "D". Then write a sequence that could be used to make your comparison. You do not need to show any work.

Series	**Convergent or Divergent?**	b_n?
(a) $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$	D	$\frac{1}{n}$

We can use the fact that $\ln(n) \leq n^d$ (eventually). Using $d = 1$ and taking the reciprocal gives

$$a_n = \frac{1}{\ln(n)} \geq \frac{1}{n} = b_n \quad \text{(eventually)}.$$

(b) $\sum_{n=1}^{\infty} \frac{3 + 2\cos(n)}{n}$	D	$\frac{1}{n}$

We know that $-1 \leq \cos(n) \leq 1$ for all n. Then $-2 \leq 2\cos(n) \leq 2$ and $1 \leq 3 + 2\cos(n) \leq 5$. Then we get the right comparison from

$$a_n = \frac{3 + 2\cos(n)}{n} \geq \frac{1}{n} = b_n$$

(c) $\sum_{n=1}^{\infty} \frac{4^n}{2^n + 8^n}$	C	$\left(\frac{1}{2}\right)^n$

From algebra (bigger denominator means smaller fraction) we know that $\frac{4^n}{2^n + 8^n} \leq \frac{4^n}{2^n}$ and $\frac{4^n}{2^n + 8^n} \leq \frac{4^n}{8^n}$ for all n. Only one of these sequences gives us a convergent series and that is

$$\frac{4^n}{8^n} = \left(\frac{1}{2}\right)^n.$$

So

$$a_n = \frac{4^n}{2^n + 8^n} \leq \left(\frac{1}{2}\right)^n = b_n.$$

Answer the following questions about the series by filling in the blanks. For "Convergent or Divergent" write "Convergent" or "C" if the series to the left is convergent. Otherwise write "Divergent" or "D". For "Test Used" write an appropriate series test that tells you whether the series is convergent or divergent. You can use test abbreviations. You do not need to show any work, although you may find it helpful to write the sums using series notation.

Series	**Convergent or Divergent?**	**Test Used?**
(d) $8-2+\frac{1}{2}-\frac{1}{8}+\frac{1}{32}-\ldots$	C	GST

Testing ratios of successive terms shows that $r=-\frac{1}{4}$ and this is the geometric series

$$\sum_{n=1}^{\infty} 8\cdot\left(-\frac{1}{4}\right)^{n-1}.$$

Since $|r|=\frac{1}{4}<1$ the series is convergent by the Geometric Series Test (GST). The Alternating Series Test (AST) would also work here.

(e) $2+\frac{3}{2}+\frac{4}{3}+\frac{5}{4}+\frac{6}{5}+\frac{7}{6}+\ldots$	D	TFD

We can recognize that the numerator and denominator each increase by one for each successive term. So this is the series

$$\sum_{n=1}^{\infty} \frac{n+1}{n}.$$

Since $\lim_{n\to\infty}\frac{n+1}{n}=1\neq 0$ the series is divergent by the Test For Divergence (TFD). The Integral Test (IT) would also work here.

(f) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{2}+\frac{1}{\sqrt{5}}+\ldots$	D	PST

This series is fairly easy to recognize as the P-series

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

where $p=\frac{1}{2}$. Since $p=\frac{1}{2}\leq 1$ the series is divergent by the P-Series Test (PST). The Integral Test (IT) would also work here.

2. Find the indicated Taylor Series. Simplify your answer.

(i) $f(x) = \ln(x)$; $\qquad a = 4$

Let $t = x - 4 \Leftrightarrow t + 4 = x$

$$\ln(x) = \ln(t+4) = \ln\left(4\left(1 - \left(-\frac{t}{4}\right)\right)\right) = \ln(4) + \ln\left(1 - \left(-\frac{t}{4}\right)\right) = \ln(4) - \sum_{n=0}^{\infty} \frac{\left(-\frac{t}{4}\right)^{n+1}}{(n+1)} =$$

$$\ln(4) + \sum_{n=0}^{\infty} \frac{(-1)^n t^{n+1}}{4^{n+1}(n+1)} = \ln(4) + \sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^{n+1}}{4^{n+1}(n+1)}.$$

(ii) $f(x) = \sin(2x)$; $\qquad a = \frac{\pi}{2}$

Let $t = x - \frac{\pi}{2} \Leftrightarrow t + \frac{\pi}{2} = x$

$$\sin(2x) = \sin\left(2\left(t + \frac{\pi}{2}\right)\right) = \sin(2t + \pi) = \sin(2t)\cos(\pi) + \sin(\pi)\cos(2t) = -\sin(2t)$$

$$-\sin(2t) = -\sum_{n=0}^{\infty} \frac{(-1)^n (2t)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1} t^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 2^{2n+1} \left(x - \frac{\pi}{2}\right)^{2n+1}}{(2n+1)!}.$$

3. (a) Determine if the series $\sum_{n=1}^{\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n$ is absolutely convergent, conditionally convergent, or divergent. Use the Root Test:

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{3n^2-3}{1+4n^2}\right)^n\right|} = \lim_{n\to\infty} \frac{3n^2-3}{1+4n^2} = \frac{3}{4} < 1$$

By the Root Test (**RoT**), the series $\sum_{n=1}^{\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n$ is absolutely convergent.

(b) What can you say about $\lim_{n\to\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n$? Support your answer with words.

We must have $\lim_{n\to\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n = 0$. If not, $\lim_{n\to\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n \neq 0$ would imply the series $\sum_{n=1}^{\infty} \left(\frac{3n^2-3}{1+4n^2}\right)^n$ is Divergent by the Test For Divergence (**TFD**). But by part (a), we know the series is absolutely convergent (and therefore convergent).

4. (a) Use techniques of integration (not power series) to evaluate the integral $\displaystyle\int \frac{\cos(6\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$.

Note that $\displaystyle\int \frac{\cos(6\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = \int \frac{\cos(6x^{1/2})}{x^{1/2}}\,\mathrm{d}x$. Use the substitution $u = 6\sqrt{x} = 6x^{1/2}$.

Then $\displaystyle\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^{-1/2} = 3 \cdot \frac{1}{\sqrt{x}}$. So $\displaystyle\frac{1}{3}\,\mathrm{d}u = \frac{1}{\sqrt{x}}\,\mathrm{d}x$ and

$$\int \frac{\cos(6\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x = \int \frac{1}{3}\cos(u)\,\mathrm{d}u = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(6\sqrt{x}) + C.$$

(b) Now evaluate the integral $\displaystyle\int \frac{\cos(6\sqrt{x})}{\sqrt{x}}\,\mathrm{d}x$ using power series.

Use the MacLaurin Series for cosine. Since $\displaystyle\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$ we know

$$\cos(6\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(6\sqrt{x})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(6x^{1/2})^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}x^n.$$

Then

$$\frac{\cos(\sqrt{x})}{\sqrt{x}} = \frac{1}{\sqrt{x}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}x^n = x^{-1/2} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}x^{n-1/2}.$$

(Fun fact: Even though this series has a fraction in the exponent, the result still converges to the desired function. This is even true for complex values of x!)

We can now integrate the power series

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}x^{n-1/2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}6^{2n}\int x^{n-1/2}\,\mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\frac{6^{2n}}{(n+1/2)}x^{n+1/2} + C.$$

5. (a) Evaluate the integral $\int 3x^2 \cdot \arctan(x)\ \mathrm{d}x$ using power series.

Use the MacLaurin Series for arctangent. Since $\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$ we know

$$3x^2 \cdot \arctan(x) = 3x^2 \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2n+1} x^{2n+3}.$$

Then

$$\int \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2n+1} x^{2n+3}\ \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{2n+1} \int x^{2n+3}\ \mathrm{d}x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 3}{(2n+1)(2n+4)} x^{2n+4} + C.$$

(b) Now use techniques of integration (not power series) to evaluate the integral $\int 3x^2 \cdot \arctan(x)\ \mathrm{d}x$.

The product in the integral suggests we should use parts:

$$\begin{aligned}\int 3x^2 \cdot \arctan(x)\ \mathrm{d}{=} \int (x^3)' \cdot \arctan(x)\ \mathrm{d}x &= x^3 \cdot \arctan(x) - \int x^3 \cdot (\arctan(x))'\ \mathrm{d}x \\ &= x^3 \cdot \arctan(x) - \int x^3 \cdot \frac{1}{1+x^2}\ \mathrm{d}x \\ &= x^3 \cdot \arctan(x) - \int \frac{x^3}{1+x^2}\ \mathrm{d}x\end{aligned}$$

For the integral $\int \frac{x^3}{1+x^2}\ \mathrm{d}x$ use the substitution $u = x^2+1$. Since $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ we have $\frac{1}{2}\ \mathrm{d}u = x\ \mathrm{d}x$. We also know that $u = x^2 + 1 \Leftrightarrow u - 1 = x^2$. Then

$$\int \frac{x^3}{1+x^2}\ \mathrm{d}x = \int \frac{x^2}{1+x^2}\ x\ \mathrm{d}x = \frac{1}{2} \int \frac{u-1}{u}\ \mathrm{d}u = \frac{1}{2} \int 1 - \frac{1}{u}\ \mathrm{d}u.$$

Then it follows that

$$\int \frac{x^3}{1+x^2}\ \mathrm{d}x = \frac{1}{2}\left(u - \ln|u|\right) + C = \frac{1}{2}\left(x^2 + 1 - \ln|x^2+1|\right) + C.$$

So we get

$$\int 3x^2 \cdot \arctan(x)\ \mathrm{d}x = x^3 \cdot \arctan(x) - \frac{1}{2}\left(x^2 + 1 - \ln|x^2+1|\right) + C.$$

6. (a) Find a range of x values for which the function is defined, then state the radius of convergence.
$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(3x+1)^n}{n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(3x+1)^{n+1}}{n+1}\cdot\frac{n}{(-1)^n(3x+1)^n}\right| = |3x+1|\cdot\lim_{n\to\infty}\frac{n}{n+1} = |3x+1|$$

Setting $|3x+1| < 1 \Rightarrow -1 < 3x+1 < 1 \Rightarrow -2 < 3x < 0 \Rightarrow -\frac{2}{3} < x < 0$

Since the center is $a = -\frac{1}{3}$ we must have $R = \frac{1}{3}$.

(b) Is the input $x = 0$ valid in the function $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(3x+1)^n}{n}$? Explain your answer.

For $x = 0$ in the series $\sum_{n=1}^{\infty} \frac{(-1)^n(3x+1)^n}{n}$ we have

$$\sum_{n=1}^{\infty} \frac{(-1)^n 1^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Here $b_n = \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$ and b_n is decreasing we know $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by the Alternating Series Test (AST). So the interval of convergence I includes $x = 0$. In other words, $x = 0$ is valid input into the function $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n(3x+1)^n}{n}$.