Math 105 C Fall 2013 NAME (legibly!) suggested solutions!

Final Exam December 10, 2013

DO NOT WRITE HERE!

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Total:	

Read the questions CAREFULLY.

Show your work in the space provided.

Make clear what your answers are.

BE NEAT.

Good Luck!

1A. What is the definition (involving limits) of the derivative of a function $f(x)$ at a point a?

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}, \text{ provided this limit exists.}$$

1B. Now let $\boxed{f(x) = x^2 + 5x + 12}$ Use the answer to 1A to find $f'(a)$. (You can of course check your work because you know how to obtain $f'(a)$ using various "rules").

$$\lim_{h\to 0} \frac{(a+h)^2 + 5(a+h) + 12 - (a^2+5a+12)}{h}, \text{ provided...}$$

$$= \lim_{h\to 0} \frac{\cancel{a^2} + 2ah + h^2 + \cancel{5a} + 5h + \cancel{12} - \cancel{a^2} - \cancel{5a} - \cancel{12}}{h}$$

$$= \lim_{h\to 0} \frac{2ah + h^2 + 5h}{h} = \lim_{h\to 0} 2a + h + 5 = 2a + 0 + 5$$

$$= \boxed{2a+5}$$

2A. *In this problem*, let $\boxed{f(x) = \sqrt{x}}$

2B. What is $f'(x)$ by the usual power rule?

writing $f(x) = x^{1/2}$ we have $f'(x) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}} = \boxed{\frac{1}{2\sqrt{x}}}$

2C. What is $f'(4)$?

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{2\cdot 2} = \boxed{\frac{1}{4} = 0.25}$$

2D.) When $f'(a)$ exists, we saw that for the same value of h, $\dfrac{f(a+h)-f(a-h)}{2h}$ usually gives better approximation to $f'(a)$ than does $\dfrac{f(a+h)-f(a)}{h}$. Use $h = 0.1$ in each of these two approximation formulas to approximate $f'(4)$. How close is each approximation to the exact value found in 2C? (find the absolute value of the difference between the exact answer and each approximation). *Is* your approximation using the first formula closer to the exact value than the second approximation? Organize your results neatly.

with $a = 4$ & $h = 0.1$ we get

$$\frac{f(a+h)-f(a-h)}{2h} = \frac{f(4.1)-f(3.9)}{0.2} = \frac{2.024845673\ldots - 1.974841766\ldots}{0.2} = \frac{0.0500039073186\ldots}{0.2}$$

$= 0.2500195366$; the error is $|0.2500195366 - 0.25| = 0.0000195366\ldots$

$$\frac{f(a+h)-f(a)}{h} = \frac{f(4.1)-f(4)}{0.1} = \frac{2.024845673-2}{0.1} = 0.24845673\ldots;$$

the error is $|0.2484567 - 0.25| = 0.00154327$

This error is much smaller, illustrating that $\frac{f(a+h)-f(a-h)}{2h}$ is the better approx. for a given h.

2E.) What is the equation of the line tangent to the graph of f at $(4, 2)$?

using $y - y_0 = m(x - x_0)$ we have

$\boxed{y - 2 = \frac{1}{4}(x-4)}$, or

$\boxed{y = \frac{1}{4}x + 1}$

3. Find the derivative y' in each problem below. You might need logarithmic differentiation and/or implicit differentiation.

3A. $y = x^{2e} + e^{3x} + \sin(4x) + 5^6 + \frac{7}{x} + \log_8(9x) + \arcsin(10) + 11^{12x}$

$$y' = \boxed{2e\,x^{2e-1} + 3e^{3x} + \cos(4x)\cdot 4 + 0 - \frac{7}{x^2} + \frac{1}{\ln 8}\cdot\frac{1}{9x}\cdot 9 + 0 + \ln(11)\cdot 11^{12x}\cdot 12}$$

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(these constants appear b/c of Chain rules at CR's)

$$\left(\text{note: } \left(\frac{7}{x}\right)' = (7x^{-1})' = -7x^{-2}\right)$$

$$\left[\text{alt: } (\log_8(9x))' = (\log_8(x) + \overbrace{\log_8(9)}^{\text{constant}})' = (\log_8(x))' = \frac{1}{(\ln 8)(x)}\right]$$

3B. $y = \arctan\left(\ln(5x^4 + x^2)\right)$

$$\left(\text{recall: } \frac{d}{dx}(\arctan u) = \left(\frac{1}{1+u^2}\right)\left(\frac{du}{dx}\right)\right)$$

$$y' = \frac{1}{1+(\ln(5x^4+x^2))^2}\cdot\left(\ln(5x^4+x^2)\right)'$$

$$= (\quad '' \quad)\cdot\left(\frac{1}{5x^4+x^2}\right)\cdot(5x^4+x^2)'$$

$$= (\quad '' \quad)\cdot(\quad '' \quad)\cdot(20x^3+2x)$$

essential!

3C. $y = \frac{x^2+x}{e^{4x}}$

Chain rule

$$y' = \frac{e^{4x}(x^2+x)' - (e^{4x})'(x^2+x)}{(e^{4x})^2} = \frac{e^{4x}(2x+1) - (4e^{4x})(x^2+x)}{(e^{4x})^2}$$

3D. $y = (x^3+x^2)^{(x^4+5)}$

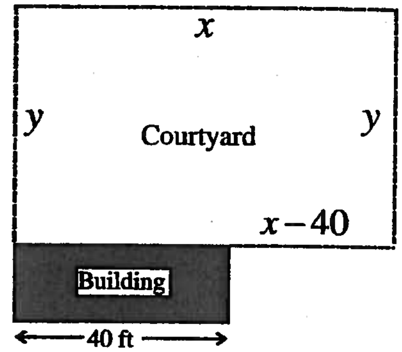
requires a product rule

$$\text{so } \ln y = \ln\left((x^3+x^2)^{(x^4+5)}\right) = (x^4+5)\cdot\ln(x^3+x^2)$$

$$\therefore \underbrace{\frac{1}{y}\cdot y'}_{\text{chain rule!}} = (4x^3)\cdot\ln(x^3+x^2) + (x^4+5)\cdot\frac{1}{x^3+x^2}\cdot(3x^2+2x)$$

$$\text{finally } y' = \left[(4x^3)\cdot\ln(x^3+x^2) + (x^4+5)\frac{1}{x^3+x^2}\cdot(3x^2+2x)\right]\underbrace{(x^3+x^2)^{(x^4+5)}}_{y}$$

4. A courtyard is to be fenced in as in the accompanying figure, where the dashed line represents the fence. The courtyard must enclose an area of 6400 square feet. There is no fence along the building. What must x and y be in order to minimize the total length of fencing required? And what will that total length of fence be? Show all your work and label your answers clearly.



minimize L = total fence length

$= y + y + x + x - 40$

$= 2y + 2x - 40$

subject to the constraint that $xy = 6400$ = total area

So $xy = 6400$ means $y = \frac{6400}{x}$; thus

$L = 2\left(\frac{6400}{x}\right) + 2x - 40$, we need stationary points: (note: $\left(\frac{6400}{x}\right)' = (6400x^{-1})' = -6400x^{-2}$)

$L' = 2 \cdot -6400x^{-2} + 2 + 0$. Set $L' = 0$ to obtain

$0 = 2(-6400)x^{-2} + 2$; (multiply by $x^2/2$:) $0 = -6400 + x^2$

$x^2 = 6400$

$\boxed{x = 80}$ ($x = -80$ doesn't count)

$y = \frac{6400}{80} = 80$;

$L = 2 \cdot 80 + 2 \cdot 80 - 40$

$= \boxed{280 \text{ (feet)}}$

5. Consider $\lim_{x \to 0} \frac{1 - \cos 4x}{3x^2}$.

5A. What "indeterminate" form does this limit have?

"0/0" b/c both numerator & denominator are → 0

5B. Show how to use l'Hôpital's rule to find this limit. (Is one "application" enough?)

$$\lim_{x \to 0} \frac{1 - \cos 4x}{3x^2} \overset{L'H}{=} \lim_{x \to 0} \underbrace{\frac{(\sin 4x)\, 4}{6x}}_{\text{still has the form "0/0"}} \overset{L'H}{=} \lim_{x \to 0} \frac{(\cos 4x) \cdot 4 \cdot 4}{6} = \frac{1 \cdot 4 \cdot 4}{6} = \boxed{\frac{16}{6}} = \boxed{8/3}$$

not indeterminate!

(in this problem $\cos 4x$ means $\cos(4x)$)

5C. Make a table that suggests what $\lim_{x \to 0} \frac{1 - \cos 4x}{3x^2}$ is: Find $\frac{1 - \cos 4x}{3x^2}$ at each of $x = 0.1$, 0.01 and 0.001. Write all table entries to at least six places after the decimal point.

x	$\frac{1 - \cos 4x}{3x^2}$
0.1	2.631300...
0.01	2.6663111...
0.001	2.66666631...
↓ 0	↓ $2.\overline{66} = 2\frac{2}{3} = 8/3$ as seen in (5B)

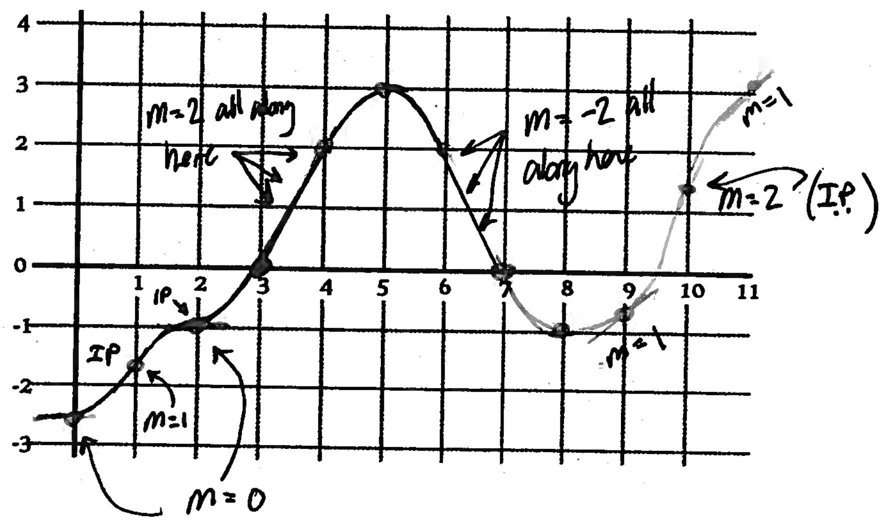
6. On the bottom grid is the graph of some function $f(t)$. It's comprised of line segments and semi-circles.

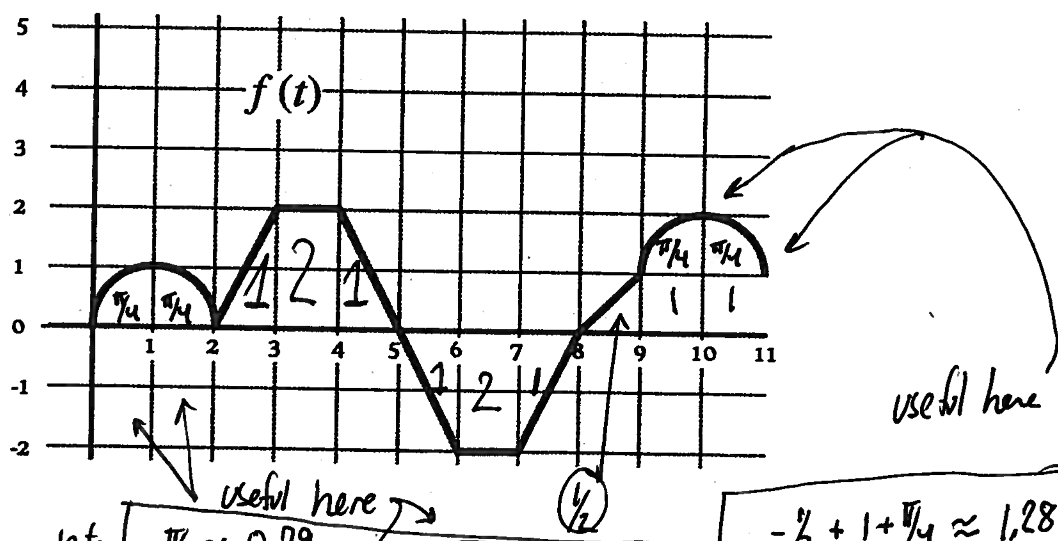
Suppose $A_f(x)$ is defined as $\int_3^x f(t)\,dt$. Make an excellent sketch of $A_f(x)$ on the top grid. Use information about $f(x)$ to make your graph of $A_f(x)$ increasing and decreasing on the intervals where it should be increasing and decreasing, and concave up and concave down where it should be; inflection points and stationary points should be obvious on your graph of $A_f(x)$.

note 1st that $A_f(3) = \int_3^3 f(t)\,dt = 0$ so the graph "starts at" $(3,0)$

Areas are found below, to find some pts on this graph.

Slopes & concavity are adjusted to match this derivative of $A_f(x)$.





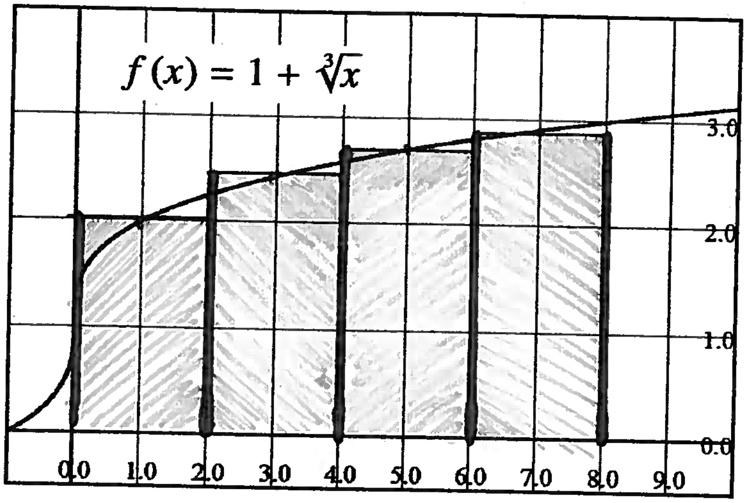
note: $\pi/4 \approx 0.79$; $2(\pi/4) = \approx 1.57$; $1 + \pi/4 = 1.79$; $1 + 2(\pi/4) = 2.57$

$-1/2 + 1 + \pi/4 \approx 1.28$ then an additional $1 + \pi/4$ gives ≈ 3.07

7. Find $\frac{d}{dx}\left(\int_3^x \cos(3t^2 + \sqrt{t})\,dt\right)$.

$$\boxed{\cos\left(3x^2 + \sqrt{x}\right)}$$

8. The function $f(x) = 1 + \sqrt[3]{x}$ is plotted below. (Remember that $\sqrt[3]{x} = x^{\frac{1}{3}}$).



8A) Find an antiderivative $F(x)$ of $f(x)$ and use it and the fundamental theorem of calculus to find the exact value of $\int_0^8 f(x)\,dx$. Show all your work.

If $f(x) = 1 + x^{1/3}$, then $F(x) = x + \dfrac{x^{4/3}}{4/3} = x + \dfrac{3}{4}x^{4/3}$ is a suitable anti-derivative

Now, $\int_0^8 f(x)\,dx = \int_0^8 (1 + x^{1/3})\,dx = x + \frac{3}{4}x^{4/3}\Big|_0^8 = F(8) - F(0)$

$= \left(8 + \frac{3}{4}8^{4/3}\right) - \left(0 + \frac{3}{4}0^{4/3}\right) = 8 + \frac{3}{4}\cdot 16 = \boxed{20}$

8B) On the graph of f, draw the four rectangles used in the MID(4) approximation of $\int_0^8 f(x)\,dx$. Then find their areas and the corresponding approximation. Show all your work (in particular, I want to see the four individual areas). Check your answer against the MID program on your calculator.

see above for the four $\square$'s. Each has base width $\Delta x = \frac{8-0}{4} = 2$.

rectangle #	1	2	3	4
midpt of base	1	3	5	7
height	$f(1)=2$	$f(3)=2.442$	$f(5)=2.70997..$	$f(7)=2.912...$
area	$\boxed{4}$	$\boxed{4.8844...}$	$\boxed{5.4199...}$	$\boxed{5.82586...}$

the sums of these areas is $\boxed{\approx 20.1303..}$

(MID(4) by calculator is 20.130134...)

8C) What is the area of the 27th rectangle as used in the L_{80} approximation of the integral?

now Δx is $\frac{8-0}{80} = 0.1$

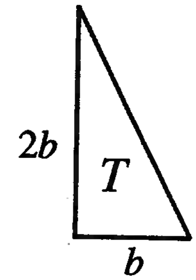
$\square$ #	1	2	3	4	...	27
left endpt	0	0.1	0.2	0.3		$26 * .1 = 2.6$

height of $\square$ #27 is $f(2.6) = 2.3750...$

area is $f(2.6) * 0.1 = \boxed{0.2375,..}$

8D) Find L_{80} and MID(80). by calculator, these sums are $\boxed{19.8871}$ and $\boxed{20.002621..}$, respectively

9. Suppose T is a right triangle which is changing in size but its height always is twice the base width (see the figure).



9A) Suppose the area of T is changing at the constant rate of 48 square inches per minute. When the base is 8 inches long, how fast is the length of the base changing?

the area A of a triangle is $A = \frac{1}{2}(\text{base})(\text{height})$

$= \frac{1}{2}(b)(2b) = b^2.$

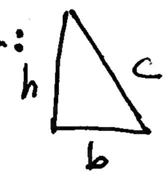
so $\boxed{\frac{dA}{dt} = 2b\frac{db}{dt}}$. we're given $\frac{dA}{dt} = 48 \frac{\text{in}^2}{\text{min}}$, $b = 8$ in, and asked for $\frac{db}{dt}$.

thus $48 \frac{\text{in}^2}{\text{min}} = 2 \cdot 8 (\text{in}) \cdot \frac{db}{dt}$; $\frac{db}{dt} = \frac{48}{16}\left(\frac{\text{in}}{\text{min}}\right) = \boxed{3 \text{ in/min}}$

9B) When the base is 8 inches long, what's the rate of change in the length of the hypotenuse?

the unneccessarily LONG way (but seems pretty common :)

let c = the length of the hypotenuse. Then from:



we get $c^2 = h^2 + b^2$

$\star$ so $2c\frac{dc}{dt} = 2h\frac{dh}{dt} + 2b\frac{db}{dt}$ and we need $\frac{dc}{dt}$.

now, $h = 2b$ so $\frac{dh}{dt} = 2\frac{db}{dt} = 2 \cdot 3 = 6 \left(\frac{\text{in}}{\text{min}}\right)$ [makes sense b/c if the height is twice the base, it "grows" twice as fast]

and $h = 2 \cdot 8 = 16$ (in)

if $c^2 = h^2 + b^2$ then $c^2 = (16)^2 + (8)^2 = 256 + 64 = 320$,

so $c = 17.888\ldots$ (inches)

Put all these "knowns" into $\star$ to get

$2 \cdot 17.88\ldots \frac{dc}{dt} = 2 \cdot 16 \cdot 6 + 2 \cdot 8 \cdot 3$ (in/min) (divide both sides by 2 to get:)

$\frac{dc}{dt} = \frac{120}{17.88\ldots} \approx 6.708\ldots$ in/min

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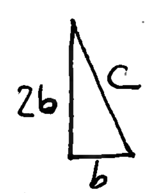
EASY WAY:

$c^2 = h^2 + b^2$

$= (2b)^2 + b^2$

$= 4b^2 + b^2$

$= 5b^2$



$\therefore c = \sqrt{5}\, b$. So $\frac{dc}{dt} = \sqrt{5}\frac{db}{dt} = \sqrt{5} \cdot 3 \frac{\text{in}}{\text{min}} = (2.236\ldots * 3) = \boxed{6.708}\left(\frac{\text{in}}{\text{min}}\right)$

10. Suppose $f(x)$ is a continuous function on the interval $[a, b]$.

10A) Suppose that k is between $f(a)$ and $f(b)$. The intermediate value theorem (IVT) guarantees there's a z in $[a, b]$ for which what equality holds?

$$f(z) = K$$

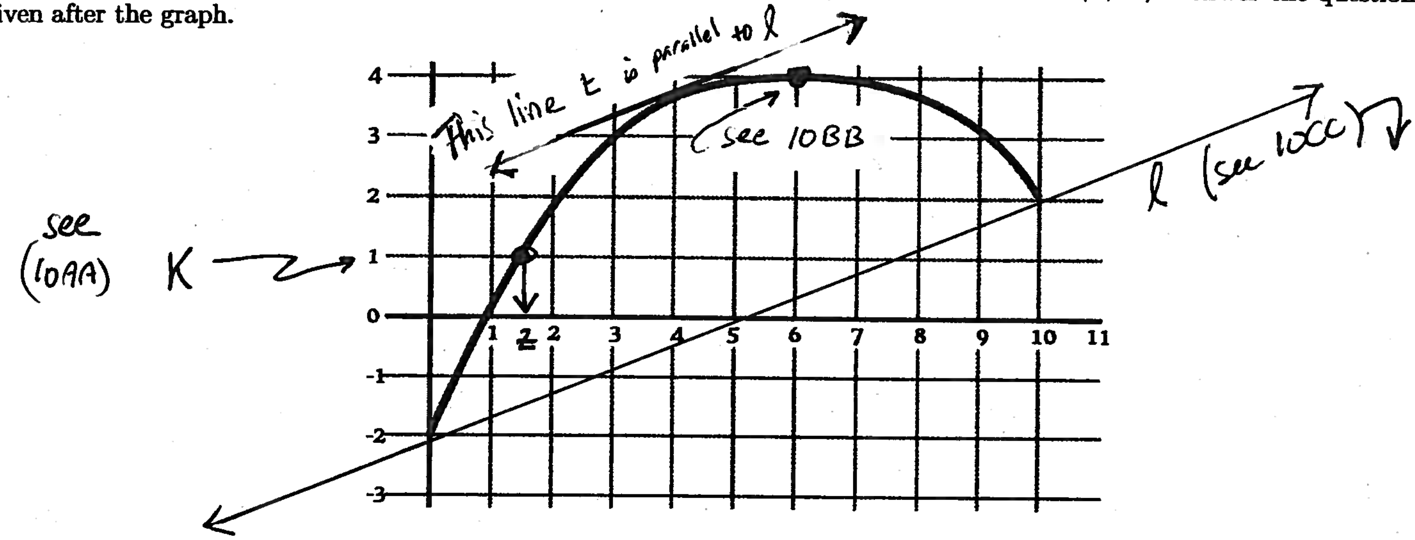
10B) The extreme value theorem (EVT) says that this function f will "attain" its global maximum somewhere on $[a, b]$. This means there's a point m in $[a, b]$ satisfying what?

$$f(m) \geq f(x) \text{ for all } x \text{ in } [a,b]$$

10C) Suppose also that f is differentiable on (a, b). The mean value theorem (MVT) says there is some c in (a, b) satisfying what?

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Now consider the following graph; it is continuous $[0, 10]$ and differentiable on the interval $(0, 10)$. Answer the questions given after the graph.



10AA) Let $k = 1$. Find the z the IVT guarantees must be there somewhere. (If there's more than one such z list them all).

$z \approx 1.5$ (note z is in $[0,10]$)

10BB) Find the m the EVT says must be there somewhere. (If there's more than one such m list them all).

$m \approx 6$ (note $m \in [0,10]$

10CC) Find the c the MVT says must be there somewhere. (If there's more than one such c list them all).

$\frac{f(b)-f(a)}{b-a}$ = avg. slope of f from a to b

= slope of line segment l added to the picture.

$f'(c)$ = slope of graph at $(c, f(c))$; the corresponding tangent line t is parallel to l, so $c \approx 4$ (since the line tangent to f at $(4, f(4))$ looks $\parallel$ to l)

(note $c \in (0, 10)$)