

Math 105: Review for Final Exam, Part II - SOLUTIONS

1. Consider the function  $f(x) = x^6 - 2x^3$  on the interval  $[-2, 2]$ .

(a) Find the  $x$ - and  $y$ -coordinates of any and all local extrema and classify each as a local maximum or local minimum.

$$\begin{aligned} f'(x) &= 6x^5 - 6x^2 \\ 0 &= 6x^2(x^3 - 1) \\ \Rightarrow x &= 0, 1 \end{aligned}$$

	$-2 \leq x < 0$	$0 < x < 1$	$1 < x \leq 2$
$f'$	negative	negative	positive
$f$	↘	↘	↗

$y$ -values:  $f(0) = 0$ ,  $f(1) = -1$

So,  $f$  has a local minimum at  $(1, -1)$ ;  $(0, 0)$  is not a local extremum.

(b) Find the  $x$ - and  $y$ -coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the  $y$ -values at the local extrema and the endpoints.

$y$ -values:  $f(-2) = 80$ ,  $f(1) = -1$ ,  $f(2) = 48$

So,  $f$  has a global minimum at  $(1, -1)$  and a global maximum at  $(-2, 80)$ .

(c) Find the  $x$ -coordinate(s) of any and all inflection points.

$$\begin{aligned} f''(x) &= 30x^4 - 12x \\ 0 &= 6x(5x^3 - 2) \\ \Rightarrow x &= 0, \sqrt[3]{0.4} \end{aligned}$$

	$x < 0$	$0 < x < \sqrt[3]{0.4}$	$\sqrt[3]{0.4} < x$
$f''$	positive	negative	positive
$f$	concave up	concave down	concave up

So, the  $x$ -values of the inflection points of  $f$  are  $x = 0$  and  $x = \sqrt[3]{0.4}$ .

2. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is \$9.00 per container, what dimensions will give the largest volume?

area of circle =  $\pi r^2$       lateral area of cylinder =  $2\pi r h$       volume of cylinder =  $\pi r^2 h$

Objective function: volume =  $V = \pi r^2 h$

We need to get this down to a function of just one variable, so we use the

constraint equation : cost =  $900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi r h$

$$900 = 6\pi r^2 + 10\pi r h$$

$$900 - 6\pi r^2 = 10\pi r h$$

$$\frac{900 - 6\pi r^2}{10\pi r} = h$$

Substituting this back into the objective function gives

$$V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10}(900r - 6\pi r^3).$$

Now that we have  $V$  as a function of just one variable, we find its maximum.

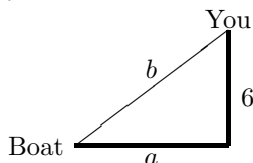
$$\begin{aligned} V'(x) &= \frac{1}{10}(900 - 18\pi r^2) \\ 0 &= \frac{1}{10}(900 - 18\pi r^2) \\ &\Rightarrow 18\pi r^2 = 900 \\ &\Rightarrow r^2 = \frac{50}{\pi} \\ &\Rightarrow r = \sqrt{\frac{50}{\pi}} \end{aligned}$$

	$0 < x < \sqrt{50/\pi}$	$\sqrt{50/\pi} < x$
$f'$	positive	negative
$f$	↗	↘

Thus, we have in fact found the global maximum at  $r = \sqrt{50/\pi}$ .

And  $h = \frac{900 - 6\pi r^2}{10\pi r} = \dots$ much simplifying... $= \sqrt{\frac{72}{\pi}} \approx 4.787$  inches.

3. **You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?**



We know  $\frac{db}{dt}$ , and we want to find  $\frac{da}{dt}$ .

So, we write an equation that relates  $a$  and  $b$  and then differentiate implicitly with respect to time  $t$ .

$$\begin{aligned} a^2 + 6^2 &= b^2 \\ 2a \frac{da}{dt} + 0 &= 2b \frac{db}{dt} \\ \frac{da}{dt} &= \frac{b}{a} \frac{db}{dt} \end{aligned}$$

At the moment in question,  $b = 10$ ,  $a = 8$  (by the Pythagorean Theorem), and  $\frac{db}{dt} = -3$ .

So,  $\frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75$  feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

4. **Use the Intermediate Value Theorem to show that  $f(x) = x^3 - 2x - 1$  has a root on  $[1, 2]$ .**  
 IVT: If  $f$  is continuous on  $[a, b]$  and  $y$  is a number between  $f(a)$  and  $f(b)$ , then there is a number  $c$  between  $a$  and  $b$  such that  $f(c) = y$ .  
 For the function given above,  $f(1) = -2$  and  $f(2) = 3$ . Since 0 is a number between  $-2$  and  $3$ , the IVT says there is a number  $c$  between 1 and 2 such that  $f(c) = 0$ ; this  $c$  is the desired root.
5. **What (if anything) does the Extreme Value Theorem say about  $f(x) = x^2$  on each of the following intervals?**  
 EVT: If  $f$  is continuous on  $[a, b]$ , then  $f$  has both a maximum and a minimum on  $[a, b]$ .

(a) **[1, 4]**

$f$  has a maximum and a minimum on  $[1, 4]$

(b) **(1, 4)**

The EVT doesn't apply because  $(1, 4)$  is not a closed interval since its endpoints are not included.

6. Find the value of the constant  $c$  that the Mean Value Theorem specifies for  $f(x) = x^3 + x$  on  $[0, 3]$ .

MVT: If  $f$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ , then there is a number  $c$  between  $a$  and  $b$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

For our function, we have  $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$ .

And  $f'(x) = 3x^2 + 1$ , so  $f'(c) = 3c^2 + 1$ .

So, we solve  $3c^2 + 1 = 10$ , which means  $c = \sqrt{3}$ . (The other solution,  $x = -\sqrt{3}$ , is not in our interval  $[0, 3]$ .)

7. Water is leaking out of a tank at a decreasing rate  $r(t)$  as shown below.

time (min)	0	2	4	6	8
rate (gal/min)	15	11	8	4	3

- (a) Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.

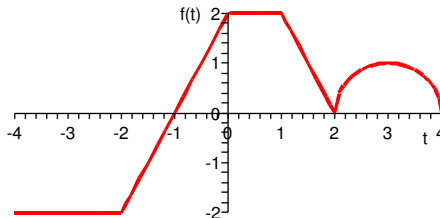
$$\text{overestimate} = L_4 = (15 + 11 + 8 + 4)(2) = 76$$

$$\text{underestimate} = R_4 = (11 + 8 + 4 + 3)(2) = 52$$

- (b) Interpret the expression  $\int_2^6 r(t) dt$  in terms of the situation described above.

This integral gives the amount (in gallons) of water that leaked from the tank on the interval  $[2, 6]$  minutes.

8. Consider the graph of  $f(t)$  shown. It is made of straight lines and a semicircle.



Let  $G(x) = \int_0^x f(t) dt$  and  $H(x) = \int_{-3}^x f(t) dt$ .

- (a) Compute  $G(2)$ ,  $G(4)$ , and  $H(4)$ .

$G(2)$  is the area under  $f$  between  $t = 0$  and  $t = 2$ . This is a rectangle plus a triangle and has area  $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$ .

Similarly,  $G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2} \pi (1)^2 = 3 + \frac{\pi}{2}$ .

$H(4)$  is the area under  $f$  between  $t = -3$  and  $t = 4$ . Remember that area below the  $t$ -axis counts as negative.

$$\begin{aligned} H(4) &= -(2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1) + \frac{1}{2} \cdot 2 \cdot 1 + [\text{area under } f \text{ from } 0 \text{ to } 4, \text{ found above as } G(4)] \\ &= -2 + \left[3 + \frac{\pi}{2}\right] \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

(b) **Where is  $G$  increasing? Where is  $G$  decreasing?**

For parts (b), (c), and (d), recall that we learned in class that  $G' = f$ .

$G$  is increasing where  $f$  is positive:  $(-1, 4]$ . Note that  $G$  has a horizontal slope at  $x = 2$  but since  $f$  is positive on each side of  $t = 2$ , we say  $G$  is increasing at  $x = 2$ .

$G$  is decreasing where  $f$  is negative:  $[-4, -1)$ .

(c) **Where is  $G$  concave up? Where is  $G$  concave down?**

$G$  is concave up where  $f$  is increasing:  $(-2, 0) \cup (2, 3)$ .

$G$  is concave down where  $f$  is decreasing:  $(1, 2) \cup (3, 4]$ .

(d) **At what  $x$ -value(s) does  $G$  have a local maximum? At what  $x$ -value(s) does  $G$  have a local minimum?**

$G$  has a local maximum where  $f$  changes from positive to negative: never.

$G$  has a local minimum where  $f$  changes from negative to positive:  $x = -1$ .

(e) **Find a formula that relates  $G$  and  $H$ .**

From their definitions,  $H(x) = \int_{-3}^0 f(t) dt + G(x) = -2 + G(x)$ .

(f) **How would your answers to (b), (c), and (d) change if the questions were about  $H$  instead of  $G$ ?**

They would not change at all because  $H'(x) = G'(x)$ .

9. (a) **Use sigma notation to express  $L_{10}$  and  $M_{10}$  as approximations to  $\int_{20}^{60} \ln x dx$ .**

We're subdividing the interval into 10 pieces, so each piece has width  $\Delta x = \frac{60 - 20}{10} = 4$ .

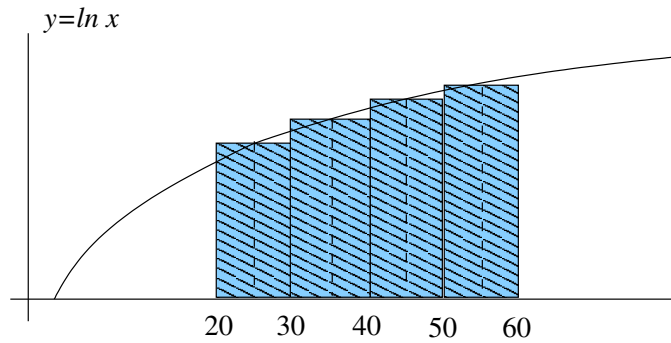
$$\begin{aligned} L_{10} &= [f(20) + f(24) + f(28) + \dots + f(52) + f(56)]\Delta x \\ &= [\ln(20) + \ln(24) + \ln(28) + \dots + \ln(52) + \ln(56)] \cdot 4 \\ &= \sum_{k=0}^9 \ln(20 + 4k) \cdot 4 \end{aligned}$$

$$\begin{aligned} M_{10} &= [f(22) + f(26) + f(30) + \dots + f(54) + f(58)]\Delta x \\ &= [\ln(22) + \ln(26) + \ln(30) + \dots + \ln(54) + \ln(58)] \cdot 4 \\ &= \sum_{k=0}^9 \ln(22 + 4k) \cdot 4 \end{aligned}$$

(b) **Draw a sketch that represents the sum  $M_4$ .**

Now we're subdividing the interval into 4 pieces, so each piece has width  $\Delta x = \frac{60 - 20}{4} = 10$ .

Note that the height of each rectangle is determined by the  $y$ -value of the curve at the *middle*  $x$ -value of the rectangle (that is, at  $x = 25, 35, 45, 55$ ).



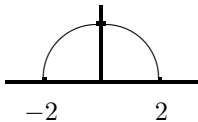
10. Find the following.

- (a) all antiderivatives of  $1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5}$

Any such antiderivative will take the form  $x + x^2 + \frac{x^4}{4} + 4\frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + C$ .

Note that we have used the facts that  $\sqrt{x} = x^{1/2}$  and  $1/x^5 = x^{-5}$ .

- (b)  $\int_{-2}^2 \sqrt{4-x^2} dx = \frac{1}{2}\pi(2)^2 = 2\pi$       This integral represents the area of a semicircle of radius 2.



- (c)  $\frac{d}{dx} \int_1^x \sin \sqrt{t} dt = \sin \sqrt{x}$

The derivative of the area function is the original function.

(d)  $\int_0^2 x^2 dx$

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that  $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$ .

We will do this with a right-hand sum  $R_n$ .

We subdivide  $[0, 2]$  into  $n$  equal pieces, each of width  $\Delta x = \frac{2-0}{n} = \frac{2}{n}$ .

Thus,  $x_1 = \frac{2}{n}$ ,  $x_2 = \frac{4}{n}$ ,  $x_3 = \frac{6}{n}$ , ..., and  $x_n = \frac{2n}{n}$ .

$$\int_0^2 x^2 dx = \lim_{n \rightarrow \infty} R_n \quad \text{This is our limit definition of the definite integral.}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \quad \text{This is our definition of a right-hand sum.}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{2k}{n}\right) \frac{2}{n} \quad \text{From above, } x_k = \frac{2k}{n} \text{ and } \Delta x = \frac{2}{n}.$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n}\right)^2 \frac{2}{n} \quad \text{Our function is } f(x) = x^2.$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{4k^2}{n^2}\right) \frac{2}{n} \quad \text{We can pull out } \frac{8}{n^3} \text{ because it doesn't depend on } k.$$

$$= \lim_{n \rightarrow \infty} \frac{8}{n^3} \sum_{k=1}^n k^2 \quad \text{We apply the handy fact we were given above.}$$

$$= \lim_{n \rightarrow \infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \frac{(n+1)(2n+1)}{3}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \frac{2n^2 + 3n + 1}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} (2 + 0 + 0)$$

$$= \frac{8}{3}$$

Now check with the FTC:  $\int_0^2 x^2 dx = \frac{x^3}{3} \Big|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$ . That was slightly easier.