Math 105: Review for Final Exam, Part II - SOLUTIONS

1. **Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is \$9.00 per container, what dimensions will give the largest volume?**

   area of circle = $\boldsymbol{\pi r^2}$ lateral area of cylinder = $\boldsymbol{2\pi rh}$ volume of cylinder = $\boldsymbol{\pi r^2 h}$

   Objective function: volume $= V = \pi r^2 h$

   We need to get this down to a function of just one variable, so we use the

$$\begin{aligned} \textit{constraint equation}: \text{cost} = 900 &= 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh \\ 900 &= 6\pi r^2 + 10\pi rh \\ 900 - 6\pi r^2 &= 10\pi rh \\ \frac{900 - 6\pi r^2}{10\pi r} &= h \end{aligned}$$

   Substituting this back into the objective function gives

   $V = \pi r^2 h = \pi r^2 \cdot \dfrac{900 - 6\pi r^2}{10\pi r} = r \cdot \dfrac{900 - 6\pi r^2}{10} = \dfrac{1}{10}(900r - 6\pi r^3)$.

   Now that we have V as a function of just one variable, we find its maximum.

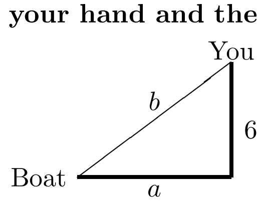
$$\begin{aligned} V'(x) &= \frac{1}{10}(900 - 18\pi r^2) \qquad \text{Since } V'(x) \text{ never fails to exist, we just need to solve } V'(x) = 0. \\ 0 &= \frac{1}{10}(900 - 18\pi r^2) \\ &\Rightarrow 18\pi r^2 = 900 \\ &\Rightarrow r^2 = \frac{50}{\pi} \\ &\Rightarrow r = \sqrt{\frac{50}{\pi}} \end{aligned}$$

	$0 < x < \sqrt{50/\pi}$	$\sqrt{50/\pi} < x$
f'	positive	negative
f	↗	↘

   Thus, we have in fact found the global maximum at $r = \sqrt{50/\pi}$.

   And $h = \dfrac{900 - 6\pi r^2}{10\pi r} =$ ...much simplifying... $= \sqrt{\dfrac{72}{\pi}} \approx 4.787$ inches.

2. **You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?**



   We know $\dfrac{db}{dt}$, and we want to find $\dfrac{da}{dt}$.

So, we write an equation that relates a and b and then differentiate implicitly with respect to time t.

$$\begin{aligned} a^2 + 6^2 &= b^2 \\ 2a\frac{da}{dt} + 0 &= 2b\frac{db}{dt} \\ \frac{da}{dt} &= \frac{b}{a}\frac{db}{dt} \end{aligned}$$

At the moment in question, $b = 10$, $a = 8$ (by the Pythagorean Theorem), and $\frac{db}{dt} = -3$.

So, $\frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75$ feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

3. **Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.**

   IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

   For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between -2 and 3, the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

4. **What (if anything) does the Extreme Value Theorem say about $f(x) = x^2$ on each of the following intervals?**

   EVT: If f is continuous on $[a, b]$, then f has both a maximum and a minimum on $[a, b]$.

   (a) **$[1, 4]$**
   f has a maximum and a minimum on $[1, 4]$

   (b) **$(1, 4)$**
   The EVT doesn't apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

5. **Find the value of the constant c that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$. [Students in the 8:00 and 9:30 sections may omit this problem.]**

   MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

   For our function, we have $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$.

   And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

   So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

6. **Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.**

time (min)	**0**	**2**	**4**	**6**	**8**
rate (gal/min)	**15**	**11**	**8**	**4**	**3**

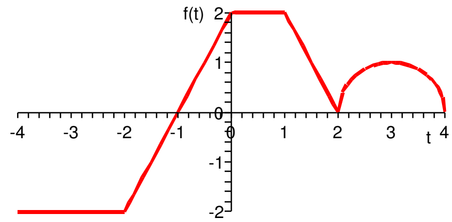
   (a) **Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.**

$$\begin{aligned} \text{overestimate} &= L_4 = (15 + 11 + 8 + 4)(2) = 76 \\ \text{underestimate} &= R_4 = (11 + 8 + 4 + 3)(2) = 52 \end{aligned}$$

   (b) **Interpret the expression $\int_2^6 r(t)\,dt$ in terms of the situation described above.**

   This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.

7. **Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.**



Let $G(x) = \int_0^x f(t)\,dt$ and $H(x) = \int_{-3}^x f(t)\,dt$.

(a) **Compute $G(2)$, $G(4)$, $G(-4)$, and $H(4)$.**

First, $G(2) = \int_0^2 f(t)\,dt$ is the area under f between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$.

Similarly, $G(4) = \int_0^4 f(t)\,dt = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 3 + \frac{\pi}{2}$.

Now, remembering that area below the t-axis counts as negative and that $\int_b^a f(t)\,dt = -\int_a^b f(t)\,dt$, we have

$G(-4) = \int_0^{-4} f(t)\,dt = -\int_{-4}^0 f(t)\,dt = -\left[-2 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2\right] = 4.$

Finally, $H(4) = \int_{-3}^4 f(t)\,dt = -2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 1 + \frac{\pi}{2}$

(b) **Where is G increasing? Where is G decreasing?**

For parts (b), (c), and (d), recall that we learned in class that $G' = f$.

G is increasing where f is positive: $(-1, 4]$. Note that G has a horizontal slope at $x = 2$ but since f is positive on each side of $t = 2$, we say G is increasing at $x = 2$.

G is decreasing where f is negative: $[-4, -1)$.

(c) **Where is G concave up? Where is G concave down?**

G is concave up where f is increasing: $(-2, 0) \cup (2, 3)$.

G is concave down where f is decreasing: $(1, 2) \cup (3, 4]$.

(d) **At what x-value(s) does G have a local maximum? At what x-value(s) does G have a local minimum?**

G has a local maximum where f changes from positive to negative: never.

G has a local minimum where f changes from negative to positive: $x = -1$.

(e) **Find a formula that relates G and H.**

From their definitions, $H(x) = \int_{-3}^0 f(t)\,dt + G(x) = -2 + G(x)$.

(f) **How would your answers to (b), (c), and (d) change if the questions were about H instead of G?**

They would not change at all because $H'(x) = G'(x)$.

8. (a) **Use sigma notation to express L_{10} and M_{10} as approximations to $\int_{20}^{60} \ln x\,dx$.**

We're subdividing the interval into 10 pieces, so each piece has width $\Delta x = \dfrac{60-20}{10} = 4$.

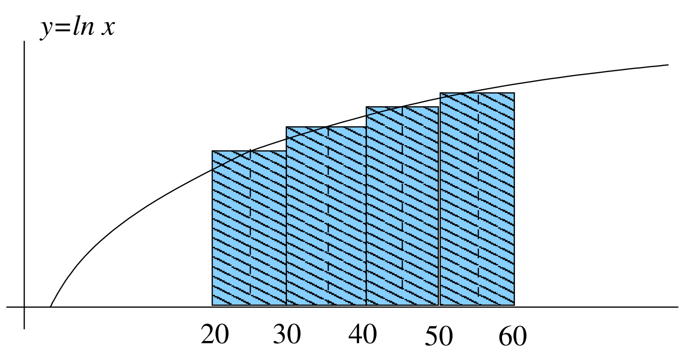
$$\begin{aligned} L_{10} &= [f(20) + f(24) + f(28) + ... + f(52) + f(56)]\Delta x \\ &= [\ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(20+4k) \cdot 4 \end{aligned}$$

$$\begin{aligned} M_{10} &= [f(22) + f(26) + f(30) + ... + f(54) + f(58)]\Delta x \\ &= [\ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(22+4k) \cdot 4 \end{aligned}$$

(b) **Draw a sketch that represents the sum M_4.**

Now we're subdividing the interval into 4 pieces, so each piece has width $\Delta x = \dfrac{60-20}{4} = 10$.

Note that the height of each rectangle is determined by the y-value of the curve at the *middle* x-value of the rectangle (that is, at $x = 25$, 35, 45, 55).



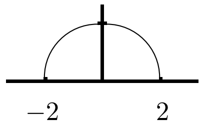
9. **Find the following.**

(a) **all antiderivatives of $1 + 2x + x^3 + 4\sqrt{x} + \dfrac{1}{x^5}$**

Any such antiderivative will take the form $x + x^2 + \dfrac{x^4}{4} + 4\dfrac{x^{3/2}}{3/2} + \dfrac{x^{-4}}{-4} + C$.

Note that we have used the facts that $\sqrt{x} = x^{1/2}$ and $1/x^5 = x^{-5}$.

(b) $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$ This integral represents the area of a semicircle of radius 2.



(c) $\displaystyle\frac{d}{dx}\int_{1}^{x} \sin\sqrt{t}\,dt = \sin\sqrt{x}$ The derivative of the area function is the original function.

(d) $\displaystyle\int_0^2 x^2\,dx$

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$.

We will do this with a right-hand sum R_n.

We subdivide $[0, 2]$ into n equal pieces, each of width $\Delta x = \dfrac{2-0}{n} = \dfrac{2}{n}$.

Thus, $x_1 = \dfrac{2}{n}$, $x_2 = \dfrac{4}{n}$, $x_3 = \dfrac{6}{n}$, ..., and $x_n = \dfrac{2n}{n}$.

$$\int_0^2 x^2\,dx = \lim_{n\to\infty} R_n \qquad \text{This is our limit definition of the definite integral.}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x \qquad \text{This is our definition of a right-hand sum.}$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f\left(\frac{2k}{n}\right)\frac{2}{n} \qquad \text{From above, } x_k = \frac{2k}{n} \text{ and } \Delta x = \frac{2}{n}.$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{2k}{n}\right)^2 \frac{2}{n} \qquad \text{Our function is } f(x) = x^2.$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{4k^2}{n^2}\right) \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{8}{n^3} \sum_{k=1}^{n} k^2 \qquad \text{We can pull out } \frac{8}{n^3} \text{ because it doesn't depend on } k.$$

$$= \lim_{n\to\infty} \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} \qquad \text{We apply the handy fact we were given above.}$$

$$= \lim_{n\to\infty} \frac{4}{n^2}\,\frac{(n+1)(2n+1)}{3}$$

$$= \lim_{n\to\infty} \frac{4}{3}\,\frac{2n^2+3n+1}{n^2}$$

$$= \lim_{n\to\infty} \frac{4}{3}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{4}{3}(2+0+0)$$

$$= \frac{8}{3}$$

Now check with the FTC: $\displaystyle\int_0^2 x^2\,dx = \left.\frac{x^3}{3}\right|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$. That was slightly easier.