Math 105: Review for Final Exam, Part I - SOLUTIONS

1. **Consider the function $f(x) = \dfrac{3}{5-2x}$.**

   (a) **Is this function continuous on the interval $(-\infty, \infty)$? Explain.**
   No. The function is discontinuous at $x = 2.5$, where f is undefined (and has a vertical asymptote).

   (b) **Compute the average rate of change of f on $[2, 2.01]$.**

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[\frac{3}{5-2(2.01)} - \frac{3}{5-2(2)}\right] \cdot \frac{1}{.01} \approx 6.122$$

   (c) **Using the limit definition of the derivative, compute $f'(x)$.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\
&= \lim_{h\to 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\
&= \lim_{h\to 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5-2(x+h)](5-2x)h} \\
&= \lim_{h\to 0} \frac{6h}{[5-2(x+h)](5-2x)h} \\
&= \lim_{h\to 0} \frac{6}{[5-2(x+h)](5-2x)} \\
&= \frac{6}{(5-2x)^2}
\end{aligned}$$

   (d) **Find the equation of the tangent line to f at $x = 2$.**
   We want $y = mx + b$. $m = f'(2) = \dfrac{6}{(5-2(2))^2} = 6$, so $y = 6x + b$.
   [Note that this slope agrees well with our answer from (b) above.]
   When $x = 2$, $y = f(2) = \dfrac{3}{5-2(2)} = 3$.
   Thus, $3 = 6 \cdot 2 + b$, so $b = -9$ and we have $y = 6x - 9$.

2. **Given that $f(0) = 2$, $g(0) = 3$, $f'(0) = 5$, $g'(0) = 7$, and $f'(3) = \pi$ compute the following.**

   (a) **$h'(0)$ if $h(z) = f(z)g(z)$**
   $h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$

   (b) **$j'(0)$ if $j(z) = \dfrac{f(z)}{g(z)}$**

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)^2]} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

   (c) **$k'(0)$ if $k(z) = f(g(z))$**
   $k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$

3. (a) **Find $\frac{dy}{dt}$ if $y = t^5 + 5^t + e^5 + \frac{t}{5} + \frac{5}{t} + \frac{5}{\sqrt[5]{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin 5$.**

$$\frac{dy}{dt} = 5t^4 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{-1}{5} t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1+(5t)^2} \cdot 5 + 0 + 0$$
$$= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{6/5}} + \frac{1}{t} + \frac{5}{1+25t^2}$$

(b) **Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x}\cos(7x^3)$.**

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}\cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3}\sin(7x^3)$$

(c) **Find $\frac{dy}{dz}$ if $y = \frac{e^z + e^\pi}{\tan 4 - 7z}$.**

$$\frac{dy}{dx} = \frac{e^z(\tan 4 - 7z) - (-7)(e^z + e^\pi)}{(\tan 4 - 7z)^2}$$

(d) **Find $\frac{dy}{dr}$ if $y = \tan\left(e^{r^2 \arcsin(5r)}\right)$.**

$$\frac{dy}{dr} = \sec^2\left(e^{r^2 \arcsin(5r)}\right) \cdot e^{r^2 \arcsin(5r)} \cdot \left[r^2 \frac{1}{\sqrt{1-25r^2}} \cdot 5 + 2r\arcsin(5r)\right]$$

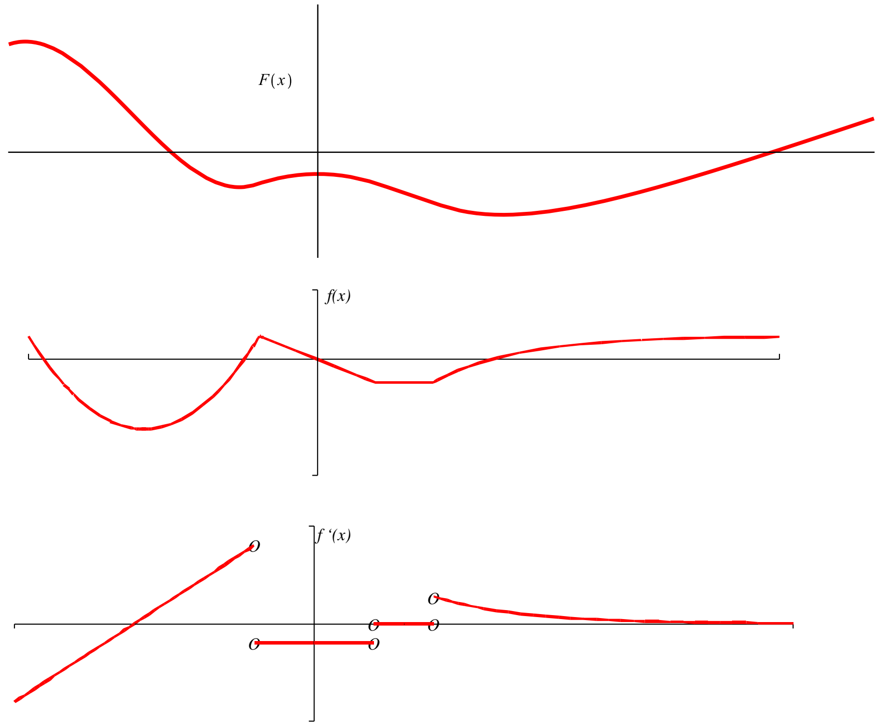
(e) **Find $\frac{dy}{dx}$ if $y^3 + yx^2 + x^2 = 3y^2$.** Here we use implicit differentiation.

$$\begin{aligned} 3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y\frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y} \end{aligned}$$

(f) **Find $\frac{dy}{dx}$ if $y = (1 + x^6)^{8x}$.** Since we have x in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned} \ln y &= 8x\ln(1 + x^6) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5 \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6}\right] \cdot y \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6}\right] \cdot (1 + x^6)^{8x} \end{aligned}$$

4. **Given the graph of f, sketch a graph of f' and a graph of F, an antiderivative of f such that $F(0) = -1$.**



Note: The concave up portion on the left side of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f, your graph of f' may be concave up/down there.

5. **Shown below is a graph of f' on its entire domain. The graph is NOT f.**

At which x-value(s)

(a) **does f have a stationary point?** c, f, h

(b) **does f have a local max?** c, h

(c) **does f have a local min?** f

(d) **does f' have a stationary point?** b, d, g, i

(e) **does f' have a local max?** b, g

(f) **does f' have a local min?** d, i

(g) **is f greatest?** c

(h) **is f least?** j

(i) **is f' greatest?** b

(j) **is f' least?** d

(k) **is f'' greatest?** e

(l) **is f'' least?** c

On what interval(s) is

(a) **f increasing?** $[a, c) \cup (f, h)$

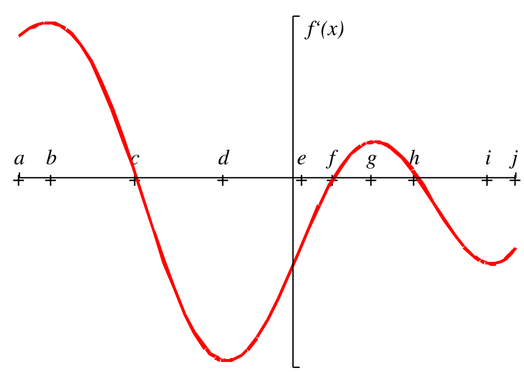
(b) **f decreasing?** $(c, f) \cup (h, j]$

(c) **f' increasing?** $[a, b) \cup (d, g) \cup (i, j]$

(d) **f' decreasing?** $(b, d) \cup (g, i)$

(e) **f concave up?** $[a, b) \cup (d, g) \cup (i, j]$

(f) **f concave down?** $(b, d) \cup (g, i)$



6. **Is $y = 7e^{3t}$ a solution to the differential equation $y'' + 2y' - 15y = 0$? Explain.**

A given function y will be a solution to the differential equation if, when we substitute in y'', y', and y, the equation is satisfied (that is, both sides of it are equal).

Since $y = 7e^{3t}$, we know that $y' = 21e^{3t}$ and $y'' = 63e^{3t}$ from the Chain Rule.

Now we check to see whether our y satisfies the differential equation.

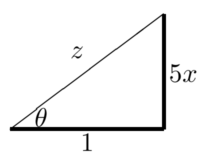
$$\begin{aligned} y'' + 2y' - 15y &\stackrel{?}{=} 0 \\ 63e^{3t} + 2 \cdot 21e^{3t} - 15 \cdot 7e^{3t} &\stackrel{?}{=} 0 \\ 63e^{3t} + 42e^{3t} - 105e^{3t} &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

So, we see that $y = 7e^{3x}$ is in fact a solution to this differential equation.

7. **Rewrite $\sin(\arctan(5x))$ as an algebraic expression. [Students in the 8:00, 9:30, and 1:10 sections may omit this problem.]**

Let $\theta = \arctan(5x)$. That is, θ is the angle whose tangent is $5x$.

We draw a triangle for which $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{5x}{1} = 5x$.



$$1^2 + (5x)^2 = z^2 \Rightarrow z = \sqrt{1 + 25x^2}$$

$$\sin(\arctan(5x)) = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{1+25x^2}}$$

8. **Evaluate the following limits.**

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\stackrel{\text{“}0/0\text{”}}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or = “0/0” or “has the form ‘$\frac{0}{0}$’ and so, by L'Hopital's Rule, is equal to” or something else. The symbol ♡ will serve the same purpose for the indeterminate form ∞/∞.

(a) $\displaystyle \lim_{x\to\infty} \frac{x^2}{\ln x} \heartsuit \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty$

(b) $\displaystyle \lim_{z\to 0} \frac{\sin(12z) - 12z}{z^3} \bigstar \lim_{z\to 0} \frac{12\cos(12z) - 12}{3z^2} \bigstar \lim_{z\to 0} \frac{-144\sin(12z)}{6z} \bigstar \lim_{z\to 0} \frac{-1728\cos(12z)}{6} = -288$

(c) $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$

(d) $\displaystyle \lim_{r\to 2} \frac{r^3 - 8}{r - 2} \bigstar \lim_{r\to 2} \frac{3r^2}{1} = 12$

(e) $\displaystyle \lim_{x\to 0^+} x^3 \ln x$ **[Students in the 8:00 and 9:30 sections may omit this problem.]**

This is of the form $0 \cdot (-\infty)$, so we rewrite it as a fraction to turn it into a L'Hopital's Rule problem.

$$\lim_{x\to 0^+} x^3 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^3}} \heartsuit \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} = \lim_{x\to 0^+} \frac{1}{x} \cdot \frac{x^4}{-3} = \lim_{x\to 0^+} \frac{x^3}{-3} = \frac{0}{-3} = 0$$