

Math 105: Review for Final Exam, Part II - SOLUTIONS

1. Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is \$9.00 per container, what dimensions will give the largest volume?

area of circle = πr^2 lateral area of cylinder = $2\pi r h$ volume of cylinder = $\pi r^2 h$

Objective function: volume = $V = \pi r^2 h$

We need to get this down to a function of just one variable, so we use the

constraint equation : cost = 900 = $3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi r h$

$$900 = 6\pi r^2 + 10\pi r h$$

$$900 - 6\pi r^2 = 10\pi r h$$

$$\frac{900 - 6\pi r^2}{10\pi r} = h$$

Substituting this back into the objective function gives

$$V = \pi r^2 h = \pi r^2 \cdot \frac{900 - 6\pi r^2}{10\pi r} = r \cdot \frac{900 - 6\pi r^2}{10} = \frac{1}{10}(900r - 6\pi r^3).$$

Now that we have V as a function of just one variable, we find its maximum.

$$V'(x) = \frac{1}{10}(900 - 18\pi r^2) \quad \text{Since } V'(x) \text{ never fails to exist, we just need to solve } V'(x) = 0.$$

$$0 = \frac{1}{10}(900 - 18\pi r^2)$$

$$\Rightarrow 18\pi r^2 = 900$$

$$\Rightarrow r^2 = \frac{50}{\pi}$$

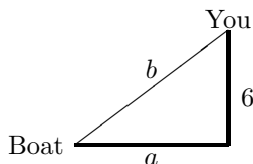
$$\Rightarrow r = \sqrt{\frac{50}{\pi}}$$

	$0 < x < \sqrt{50/\pi}$	$\sqrt{50/\pi} < x$
f'	positive	negative
f	↗	↘

Thus, we have in fact found the global maximum at $r = \sqrt{50/\pi}$.

And $h = \frac{900 - 6\pi r^2}{10\pi r} = \dots$ much simplifying... $= \sqrt{\frac{72}{\pi}} \approx 4.787$ inches.

2. You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?



We know $\frac{db}{dt}$, and we want to find $\frac{da}{dt}$.

So, we write an equation that relates a and b and then differentiate implicitly with respect to time t .

$$\begin{aligned} a^2 + 6^2 &= b^2 \\ 2a \frac{da}{dt} + 0 &= 2b \frac{db}{dt} \\ \frac{da}{dt} &= \frac{b}{a} \frac{db}{dt} \end{aligned}$$

At the moment in question, $b = 10$, $a = 8$ (by the Pythagorean Theorem), and $\frac{db}{dt} = -3$.

So, $\frac{da}{dt} = \frac{10}{8} \cdot (-3) = -3.75$ feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

3. **Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.**

IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between -2 and 3 , the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

4. **What (if anything) does the Extreme Value Theorem say about $f(x) = x^2$ on each of the following intervals?**

EVT: If f is continuous on $[a, b]$, then f has both a maximum and a minimum on $[a, b]$.

- (a) **$[1, 4]$**

f has a maximum and a minimum on $[1, 4]$

- (b) **$(1, 4)$**

The EVT doesn't apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

5. **Find the value of the constant c that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.**

MVT: If f is continuous on $[a, b]$ and differentiable on (a, b) , then there is a number c between a and b such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

For our function, we have $\frac{f(3) - f(0)}{3 - 0} = \frac{30 - 0}{3} = 10$.

And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

6. **Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.**

time (min)	0	2	4	6	8
rate (gal/min)	15	11	8	4	3

- (a) **Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.**

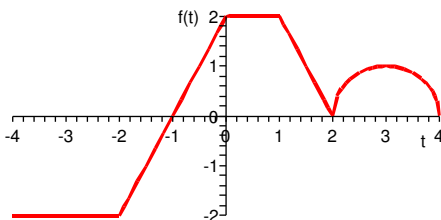
$$\text{overestimate} = L_4 = (15 + 11 + 8 + 4)(2) = 76$$

$$\text{underestimate} = R_4 = (11 + 8 + 4 + 3)(2) = 52$$

- (b) **Interpret the expression $\int_2^6 r(t) dt$ in terms of the situation described above.**

This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.

7. Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.



Let $G(x) = \int_0^x f(t) dt$ and $H(x) = \int_{-3}^x f(t) dt$.

- (a) Compute $G(2)$, $G(4)$, $G(-4)$, and $H(4)$.

First, $G(2) = \int_0^2 f(t) dt$ is the area under f between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$.

Similarly, $G(4) = \int_0^4 f(t) dt = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 3 + \frac{\pi}{2}$.

Now, remembering that area below the t -axis counts as negative and that $\int_b^a f(t) dt = -\int_a^b f(t) dt$, we have

$$G(-4) = \int_0^{-4} f(t) dt = -\int_{-4}^0 f(t) dt = -\left[-2 \cdot 2 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2\right] = 4.$$

$$\text{Finally, } H(4) = \int_{-3}^4 f(t) dt = -2 \cdot 1 - \frac{1}{2} \cdot 1 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 2 + 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 1 + \frac{\pi}{2}$$

- (b) Where is G increasing? Where is G decreasing?

For parts (b), (c), and (d), recall that we learned in class that $G' = f$.

G is increasing where f is positive: $(-1, 4]$. Note that G has a horizontal slope at $x = 2$ but since f is positive on each side of $t = 2$, we say G is increasing at $x = 2$.

G is decreasing where f is negative: $[-4, -1)$.

- (c) Where is G concave up? Where is G concave down?

G is concave up where f is increasing: $(-2, 0) \cup (2, 3)$.

G is concave down where f is decreasing: $(1, 2) \cup (3, 4]$.

- (d) At what x -value(s) does G have a local maximum? At what x -value(s) does G have a local minimum?

G has a local maximum where f changes from positive to negative: never.

G has a local minimum where f changes from negative to positive: $x = -1$.

- (e) Find a formula that relates G and H .

$$\text{From their definitions, } H(x) = \int_{-3}^0 f(t) dt + G(x) = -2 + G(x).$$

- (f) How would your answers to (b), (c), and (d) change if the questions were about H instead of G ?

They would not change at all because $H'(x) = G'(x)$.

8. (a) Use sigma notation to express L_{10} and M_{10} as approximations to $\int_{20}^{60} \ln x dx$.

We're subdividing the interval into 10 pieces, so each piece has width $\Delta x = \frac{60 - 20}{10} = 4$.

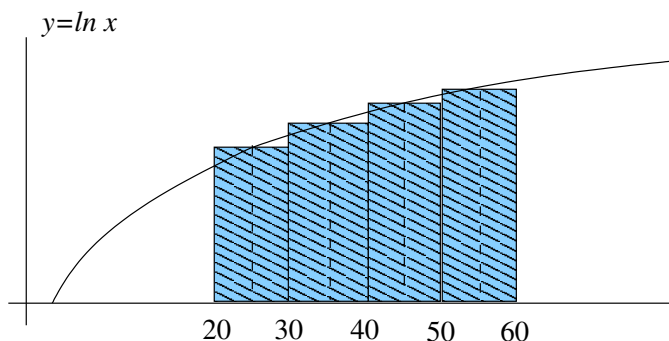
$$\begin{aligned} L_{10} &= [f(20) + f(24) + f(28) + \dots + f(52) + f(56)]\Delta x \\ &= [\ln(20) + \ln(24) + \ln(28) + \dots + \ln(52) + \ln(56)] \cdot 4 \\ &= \sum_{k=0}^9 \ln(20 + 4k) \cdot 4 \end{aligned}$$

$$\begin{aligned} M_{10} &= [f(22) + f(26) + f(30) + \dots + f(54) + f(58)]\Delta x \\ &= [\ln(22) + \ln(26) + \ln(30) + \dots + \ln(54) + \ln(58)] \cdot 4 \\ &= \sum_{k=0}^9 \ln(22 + 4k) \cdot 4 \end{aligned}$$

(b) **Draw a sketch that represents the sum M_4 .**

Now we're subdividing the interval into 4 pieces, so each piece has width $\Delta x = \frac{60 - 20}{4} = 10$.

Note that the height of each rectangle is determined by the y -value of the curve at the *middle* x -value of the rectangle (that is, at $x = 25, 35, 45, 55$).



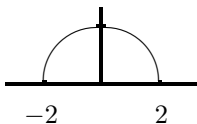
9. Find the following.

(a) all antiderivatives of $1 + 2x + x^3 + 4\sqrt{x} + \frac{1}{x^5} + \sec^2(6x) + \frac{7}{1 + 100x^2}$

Any such antiderivative will take the form $x + x^2 + \frac{x^4}{4} + 4\frac{x^{3/2}}{3/2} + \frac{x^{-4}}{-4} + \frac{\tan(6x)}{6} + \frac{7 \arctan(10x)}{10} + C$.

Note that we have used the facts that $\sqrt{x} = x^{1/2}$ and $1/x^5 = x^{-5}$.

(b) $\int_{-2}^2 \sqrt{4 - x^2} dx = \frac{1}{2}\pi(2)^2 = 2\pi$ This integral represents the area of a semicircle of radius 2.



(c) $\frac{d}{dx} \int_1^x \sin \sqrt{t} dt = \sin \sqrt{x}$

The derivative of the area function is the original function.

(d) $\int_0^2 x^2 dx$

Do this first with the limit definition of the definite integral then check your answer with the Fundamental Theorem.

You may use the fact that $\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$.

We will do this with a right-hand sum R_n .

We subdivide $[0, 2]$ into n equal pieces, each of width $\Delta x = \frac{2-0}{n} = \frac{2}{n}$.

Thus, $x_1 = \frac{2}{n}$, $x_2 = \frac{4}{n}$, $x_3 = \frac{6}{n}$, ..., and $x_n = \frac{2n}{n}$.

$$\int_0^2 x^2 dx = \lim_{n \rightarrow \infty} R_n \quad \text{This is our limit definition of the definite integral.}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \quad \text{This is our definition of a right-hand sum.}$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n f\left(\frac{2k}{n}\right) \frac{2}{n} \quad \text{From above, } x_k = \frac{2k}{n} \text{ and } \Delta x = \frac{2}{n}.$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{2k}{n}\right)^2 \frac{2}{n} \quad \text{Our function is } f(x) = x^2.$$

$$= \lim_{n \rightarrow \infty} \sum_{k=1}^n \left(\frac{4k^2}{n^2}\right) \frac{2}{n} \quad \text{We can pull out } \frac{8}{n^3} \text{ because it doesn't depend on } k.$$

$$= \lim_{n \rightarrow \infty} \frac{8}{n^3} \sum_{k=1}^n k^2 \quad \text{We apply the handy fact we were given above.}$$

$$= \lim_{n \rightarrow \infty} \frac{8}{n^3} \frac{n(n+1)(2n+1)}{6}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{n^2} \frac{(n+1)(2n+1)}{3}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \frac{2n^2 + 3n + 1}{n^2}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} \left(2 + \frac{3}{n} + \frac{1}{n^2}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{4}{3} (2 + 0 + 0)$$

$$= \frac{8}{3}$$

Now check with the FTC: $\int_0^2 x^2 dx = \frac{x^3}{3} \Big|_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}$. That was slightly easier.