

Math 105: Review for Final Exam, Part I - SOLUTIONS

1. Consider the function $f(x) = \frac{3}{5-2x}$.

(a) Is this function continuous on the interval $(-\infty, \infty)$? Explain.

No. f is discontinuous at $x = 2.5$, where f is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of f on $[2, 2.01]$.

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[\frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122$$

(c) Using the limit definition of the derivative, compute $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\ &= \lim_{h \rightarrow 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6h}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6}{[5 - 2(x+h)](5-2x)} \\ &= \frac{6}{(5-2x)^2} \end{aligned}$$

(d) Find the equation of the tangent line to f at $x = 2$.

We want $y = mx + b$. $m = f'(2) = \frac{6}{(5-2(2))^2} = 6$, so $y = 6x + b$.

[Note that this slope agrees well with our answer from (b) above.]

When $x = 2$, $y = f(2) = \frac{3}{5-2(2)} = 3$.

Thus, $3 = 6 \cdot 2 + b$, so $b = -9$ and we have $y = 6x - 9$.

2. Given that $f(0) = 2$, $g(0) = 3$, $f'(0) = 5$, $g'(0) = 7$, and $f'(3) = \pi$ compute the following.

(a) $h'(0)$ if $h(z) = f(z)g(z)$

$$h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$$

(b) $j'(0)$ if $j(z) = \frac{f(z)}{g(z)}$

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

(c) $k'(0)$ if $k(z) = f(g(z))$

$$k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$$

3. (a) Find $\frac{dy}{dt}$ if $y = t^5 + 5^t + e^5 + \frac{t}{5} + \frac{5}{t} + \frac{5}{\sqrt[5]{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin 5$.

$$\begin{aligned}\frac{dy}{dt} &= 5t^4 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{-1}{5}t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1+(5t)^2} \cdot 5 + 0 + 0 \\ &= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{6/5}} + \frac{1}{t} + \frac{5}{1+25t^2}\end{aligned}$$

- (b) Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x} \cos(7x^3)$.

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)$$

- (c) Find $\frac{dy}{dz}$ if $y = \frac{e^z + e^\pi}{\tan 4 - 7z}$.

$$\frac{dy}{dz} = \frac{e^z(\tan 4 - 7z) - (-7)(e^z + e^\pi)}{(\tan 4 - 7z)^2}$$

- (d) Find $\frac{dy}{dr}$ if $y = \tan(e^{r^2} \arcsin(5r))$.

$$\frac{dy}{dr} = \sec^2(e^{r^2} \arcsin(5r)) \cdot e^{r^2} \arcsin(5r) \cdot \left[r^2 \frac{1}{\sqrt{1-25r^2}} \cdot 5 + 2r \arcsin(5r) \right]$$

- (e) Find $\frac{dy}{dx}$ if $y^3 + yx^2 + x^2 = 3y^2$.

Here we use implicit differentiation.

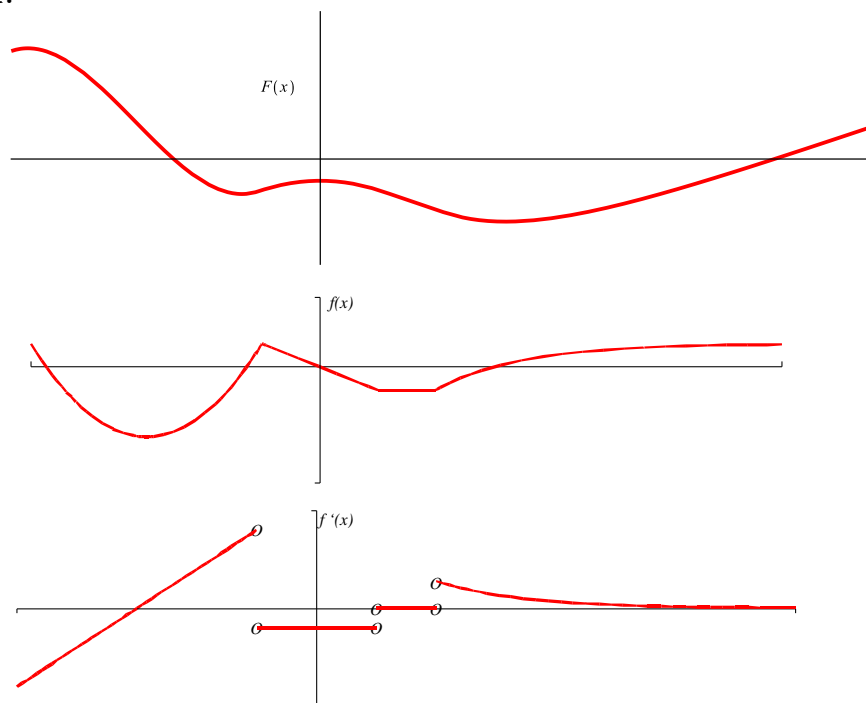
$$\begin{aligned}3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y \frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y}\end{aligned}$$

- (f) Find $\frac{dy}{dt}$ if $y = (1 + x^6)^{8x}$.

Since we have x in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned}\ln y &= 8x \ln(1 + x^6) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5 \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot y \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot (1 + x^6)^{8x}\end{aligned}$$

4. Given the graph of f , sketch a graph of f' and a graph of F , an antiderivative of f such that $F(0) = -1$.



Note: The concave up portion on the left side of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f , your graph of f' may be concave up/down there.

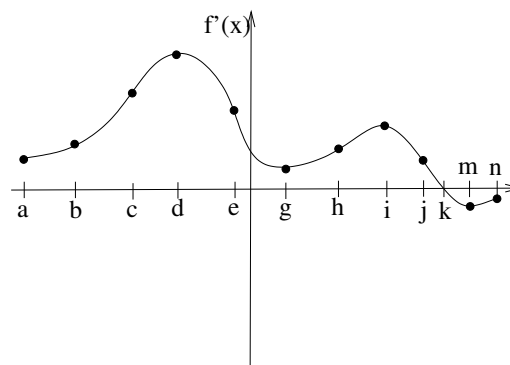
5. Shown below is a graph of f' on its entire domain. The graph is NOT f .

At which x -value(s) (if any)

- (a) does f have a stationary point? k
- (b) does f have a local max? k
- (c) does f have a local min? none
- (d) does f' have a stationary point? d, g, i, m
- (e) does f' have a local max? d, i
- (f) does f' have a local min? g, m
- (g) is f greatest? k
- (h) is f least? a
- (i) is f' greatest? d
- (j) is f' least? m
- (k) is f'' greatest? c
- (l) is f'' least? e

- (b) f decreasing? k to n
- (c) f' increasing? a to d and g to i and m to n
- (d) f' decreasing? d to g and i to m
- (e) f concave up? a to d and g to i and m to n
- (f) f concave down? d to g and i to m

* Whether to include the endpoints of these intervals will depend on your instructor's definitions.



On what interval(s)* is

- (a) f increasing? a to k

6. Solve the IVP $y' = e^x - \sin x + 5$ given that $y(0) = 3$.

We antidifferentiate each side to obtain $y(x) = e^x + \cos x + 5x + C$. To find C , we let $x = 0$, meaning $3 = e^0 + \cos 0 + 5 \cdot 0 + C$, so $C = 1$ and our solution is $y(x) = e^x + \cos x + 5x + 1$.

7. Evaluate the following limits.

Throughout this solution, the symbol \star will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form $0/0$; this may be $\stackrel{0/0}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or $=$ or "0/0" or $\text{"has the form '0/0', and so, by L'Hopital's Rule, is equal to"}$ or something else. The symbol \heartsuit will serve the same purpose for the indeterminate form ∞/∞ .

$$(a) \lim_{x \rightarrow \infty} \frac{x^2}{\ln x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{1/x} = \lim_{x \rightarrow \infty} 2x^2 = \infty$$

$$(b) \lim_{z \rightarrow 0} \frac{\sin(5z) - 5z}{z^3} \star \lim_{z \rightarrow 0} \frac{5 \cos(5z) - 5}{3z^2} \star \lim_{z \rightarrow 0} \frac{-25 \sin(5z)}{6z} \star \lim_{z \rightarrow 0} \frac{-125 \cos(12z)}{6} = -\frac{125}{6}$$

$$(c) \lim_{x \rightarrow 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$$

$$(d) \lim_{r \rightarrow 2} \frac{r^3 - 8}{r - 2} \star \lim_{r \rightarrow 2} \frac{3r^2}{1} = 12$$

8. Consider the function $f(x) = x^6 - 2x^3$ on the interval $[-2, 2]$.

- (a) Find the x - and y -coordinates of any and all critical points and classify each as a local maximum, local minimum, or neither.

$$\begin{aligned} f'(x) &= 6x^5 - 6x^2 && \text{Since } f'(x) \text{ never fails to exist, we just need to solve } f'(x) = 0. \\ 0 &= 6x^2(x^3 - 1) \\ \Rightarrow x &= 0, 1 \end{aligned}$$

| | | | |
|------|-----------------|-------------|----------------|
| | $-2 \leq x < 0$ | $0 < x < 1$ | $1 < x \leq 2$ |
| f' | negative | negative | positive |
| f | ↘ | ↘ | ↗ |

$$y\text{-values: } f(0) = 0, f(1) = -1$$

So, f has a local minimum at $(1, -1)$; $(0, 0)$ is not a local extremum.

- (b) Find the x - and y -coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the y -values at the local extrema and the endpoints.

$$y\text{-values: } f(-2) = 80, f(1) = -1, f(2) = 48$$

So, f has a global minimum at $(1, -1)$ and a global maximum at $(-2, 80)$.

- (c) Find the x -coordinate(s) of any and all inflection points.

$$\begin{aligned} f''(x) &= 30x^4 - 12x && \text{Since } f''(x) \text{ never fails to exist, we just need to solve } f''(x) = 0. \\ 0 &= 6x(5x^3 - 2) \\ \Rightarrow x &= 0, \sqrt[3]{0.4} \end{aligned}$$

| | | | |
|-------|------------|-------------------------|---------------------|
| | $x < 0$ | $0 < x < \sqrt[3]{0.4}$ | $\sqrt[3]{0.4} < x$ |
| f'' | positive | negative | positive |
| f | concave up | concave down | concave up |

So, the x -values of the inflection points of f are $x = 0$ and $x = \sqrt[3]{0.4}$.