1. (21 pts) Find the derivative of each function below. Simplify your answers.

   (a) \( f(x) = x^3 \cos(4x) \)

   \[
   f'(x) = 3x^2 \cos(4x) + x^3 (-\sin(4x) \cdot 4)
   = 3x^2 \cos(4x) - 4x^3 \sin(4x)
   \]

   (b) \( f(x) = \frac{2x - e^x}{e^x} \)

   \[
   f'(x) = \frac{(2 \ln 2 - e^x) e^x - (2x - e^x) \cdot e^x}{(e^x)^2}
   = \frac{2x \ln 2 - e^x - 2x + e^x}{e^x}
   = \frac{2x (\ln 2 - 1)}{e^x}
   \]

   (c) \( f(x) = \ln(e^{\sin x + x^2}) = \sin x + x^2 \)  \( \text{since} \ \ln e^u = u \)

   \[
   f'(x) = \cos x + 2x
   \]

Note: This can also be done using the chain rule:

\[
\frac{d}{dx} \ln(e^{\sin x + x^2}) \cdot (\cos x + 2x) = \cos x + 2x
\]
2. (15 pts) Consider the function \( f \) with first and second derivatives given below:

\[
\begin{align*}
  f'(x) &= \frac{5(x - 4)}{3\sqrt[3]{x}} \\
  f''(x) &= \frac{10(x + 2)}{9\sqrt[3]{x^4}}
\end{align*}
\]

(a) Find the critical points of \( f \). Classify each as a local maximum, local minimum, or neither.

Critical points occur where \( f'(x) = 0 \) or \( f'(x) \) is undefined

\[
\begin{align*}
  f'(x) = 0 \text{ when } 5(x-4) = 0 & \Rightarrow x = 4 \\
  f'(x) \text{ is undefined when } \sqrt[3]{x} = 0 & \Rightarrow x = 0
\end{align*}
\]

Therefore, \( f \) has a local max at \( x = 0 \) and local min at \( x = 4 \)

(b) Find any inflection points of \( f \). Be sure to also consider where \( f''(x) \) is undefined when identifying possible inflection points.

Possible inflection pts occur where \( f''(x) = 0 \) or \( f'(x) \) is undefined

\[
\begin{align*}
  f''(x) = 0 \text{ when } 10(x+2) = 0 & \Rightarrow x = -2 \\
  f''(x) \text{ is undefined when } \sqrt[3]{x} = 0 & \Rightarrow x = 0
\end{align*}
\]

Therefore, \( f \) has an inflection point at \( x = -2 \)

(c) Use the information in parts (a)–(b) to sketch \( f \) so that \( f \) passes through the point \((0, 5)\). Label all critical points and inflection points.

3. (10 pts) Find the equation of the tangent line to the curve \( 2x^2y - y^2 = x \) at the point \((1, 1)\).

Use implicit differentiation to find \( \frac{dy}{dx} \)

\[
\begin{align*}
  4xy + 2x^2 \frac{dy}{dx} - 2y \frac{dy}{dx} &= 1 \\
  (2x^2 - 2y) \frac{dy}{dx} &= 1 - 4xy \\
  \frac{dy}{dx} &= \frac{1 - 4xy}{2x^2 - 2y}
\end{align*}
\]

Slope at \((1, 1)\) is \( \frac{dy}{dx}(1, 1) = \frac{1 - 4(1)(1)}{2(1)^2 - 2(1)} = \text{undefined} \)

The equation of the tangent line is the vertical line passing through \((1, 1)\) \( \Rightarrow x = 1 \)
4. (20 pts) Consider \( f(x) = \frac{e^x}{x^2 - 1} \).

(a) Using limits, complete the definition: “The line \( y = k \) is a horizontal asymptote for the graph of \( f \) if . . .”

\[
\lim_{x \to \pm \infty} f(x) = k \quad \text{or} \quad \lim_{x \to -\infty} f(x) = k
\]

(b) Find all horizontal asymptotes, if any exist, for the graph of \( f(x) \). Justify your answer using limits.

\[
\lim_{x \to \infty} \frac{e^x}{x^2 - 1} = \frac{\infty}{\infty} \quad \text{IF “\infty”} \quad \lim_{x \to \infty} \frac{e^x}{2x} = \frac{\infty}{\infty} \quad \text{IF “\infty”} \quad \lim_{x \to \infty} \frac{e^x}{2} = \infty \quad \text{so there is no horizontal asymptote on the right side of the graph.}
\]

\[
\lim_{x \to -\infty} \frac{e^x}{x^2 - 1} = 0 
\]

therefore \( y = 0 \) is a horizontal asymptote for the graph of \( f \).

(c) Using limits, complete the definition: “The line \( x = a \) is a vertical asymptote for the graph of \( f \) if . . .”

\[
\lim_{x \to a^-} f(x) = \infty \quad \text{or} \quad -\infty \quad \text{or} \quad \lim_{x \to a^+} f(x) = \infty \quad \text{or} \quad -\infty
\]

(d) Find all vertical asymptotes, if any exist, for the graph of \( f(x) \). Justify your answer using limits.

possible vertical asymptotes occur where \( f \) is undefined, i.e., where \( x = \pm 1 \) since \( f(1) \) \& \( f(-1) \) DNE

\[
\lim_{x \to 1^-} \frac{e^x}{x^2 - 1} = -\infty \quad \text{since} \quad x^2 - 1 \to 0 \text{ from the left as } x \to 1^-
\]

\[
\Rightarrow x = 1 \text{ is a vertical asymptote.}
\]

\[
\lim_{x \to 1^+} \frac{e^x}{x^2 - 1} = +\infty \quad \text{since} \quad x^2 - 1 \to 0 \text{ from the right as } x \to 1^+
\]

\[
\Rightarrow x = -1 \text{ is a vertical asymptote.}
\]
5. (15 pts) A window has the shape of a rectangle sur-
mounted by a semicircle. If the perimeter of the
window is 20 feet, find the dimensions of the rectan-
gle that will produce the largest area for the window.
Neglect the thickness of the frame. (Helpful formu-
las related to circles: Area = πr² & Circumference
= 2πr.)

(a) What quantity are you trying to optimize?
Are you trying to minimize it or maximize it?

Maximize area of window

(b) What is the objective function for the quantity you are trying to optimize?

\[
\text{area of semi-circle} = \frac{1}{2}πr^2 \quad \text{area of rectangle} = 2rh
\]

Objective Function \( A = \frac{1}{2}πr^2 + 2rh \)

(c) Find the constraint equation and use it to rewrite the objective function from (b) as a function of one
variable.

\[
\text{Perimeter} \quad 20 = πr + 2r + 2h
\]

solve for \( h \):

\[
\frac{dh}{dr} = 20 - πr - 2r
\]

\[
h = \frac{20-πr-2r}{2}
\]

\[
A(r) = 20r - \frac{1}{2}πr^2 - 2r^2
\]

(d) Find the critical point(s) of the objective function. Verify that you have the desired max or min.

\[
A'(r) = 20 - πr - 4r
\]

\[
A'(r) = 0 \text{ when } 20 = πr + 4r
\]

\[
r = \frac{20}{π+4} \approx 2.8
\]

\[
A''(r) = -π - 4 < 0 \text{ for all values of } r
\]

So \( A \) is concave down on its domain

Therefore, by the 2nd Derivative Test

\( A \) has a maximum at the critical point \( r = \frac{20}{π+4} \)

(e) What are the dimensions of the rectangle?

\[
\text{Rectangle has dimensions } (2r) \times h
\]

\[
\text{Since } r \approx 2.8 \text{ then } 2r \approx 5.6
\]

\[
\text{and } h = \frac{1}{2}(20 - π(2.8) - 2(2.8)) \approx 2.8
\]

\[
\Rightarrow 5.6 \text{ ft} \times 2.8 \text{ ft}
\]
6. (a) (7 pts) Rewrite \( f(x) = \csc(\arccos(2x)) \) as an algebraic expression - no trigonometric or inverse trigonometric functions.

\[
\text{let } \theta = \arccos(2x), \text{ then } \cos \theta = 2x = \frac{2x}{1} \quad \text{adj} \quad \text{hyp}.
\]

Therefore, \( f(x) = \csc(\arccos(2x)) \)

\[
\csc \theta = \frac{\text{hyp}}{\text{opp}} \quad \text{use triangle}
\]

\[
= \frac{1}{\sqrt{1 - 4x^2}}
\]

(b) (5 pts) Use your answer from (a) to find \( f'(x) \). Note: \( f' \) should also be an algebraic expression. Simplify your answer.

From above, \( f(x) = \frac{1}{\sqrt{1 - 4x^2}} = (1 - 4x^2)^{-\frac{1}{2}} \)

\[
f'(x) = -\frac{1}{2}(1 - 4x^2)^{-\frac{3}{2}} \cdot (-8x)
\]

\[
f'(x) = \frac{4x}{\sqrt{(1 - 4x^2)^3}}
\]

7. (7 pts) Use logarithmic differentiation to find \( \frac{dy}{dx} \) when \( y = \sqrt{x^x} \).

\[
y = \sqrt{x^x} \quad \text{take ln of both sides}
\]

\[
\ln y = \ln \sqrt{x^x}
\]

\[
\ln y = x \cdot (\ln x) \quad (\text{since } \ln a^b = b \cdot \ln a)
\]

Now differentiate both sides of the equation, then solve for \( \frac{dy}{dx} \).

\[
\frac{1}{y} \cdot \frac{dy}{dx} = 1 \cdot \ln x + x \cdot \frac{1}{\sqrt{x^x}} \cdot \frac{1}{2\sqrt{x}} \quad \text{product rule}
\]

\[
\frac{dy}{dx} = y \left( \ln x + \frac{1}{2} \right)
\]

\[
\frac{dy}{dx} = \sqrt{x^x} \left( \ln x + \frac{1}{2} \right)
\]