Name:________________________________

Math 105: Fall 2012
Exam 2: November 9

Correct answers accompanied by incorrect or incomplete work will not receive full credit.

1. (24 points) Find y' in **3 of 4** of the following. If you do more than three, then clearly mark which three you want graded. If you don't, the worst three will be chosen for you.

   (a) $y = \arctan(x^3 + 2x) + \frac{\pi}{x^3} + \ln 4$

   (b) $y = (\arcsin x)(e^{x^3}) + 2^{-x} + \cos\frac{\pi}{5}$

   (c) $y = \sqrt{x^3 + \ln(5x^4 + 3)}$

   (d) $y = x^x$

(a) $y' = \dfrac{3x^2+2}{1+(x^3+2x)^2} + (-3)\pi x^{-4}$

(b) $\left[\dfrac{e^{x^3}}{\sqrt{1-x^2}} + 3x^2e^{x^3}\arcsin x\right] + (-1)(\ln 2)2^{-x}$

(c) $y = (x^3 + \ln(5x^4+3))^{1/2}$

$y' = \frac{1}{2}(x^3 + \ln(5x^4+3))^{-1/2}\left[3x^2 + \dfrac{20x^3}{5x^4+3}\right]$

(d) $y = x^x$

$\ln y = \ln x^x = x\ln x$

$\dfrac{1}{y}\dfrac{dy}{dx} = x \cdot \frac{1}{x} + \ln x = 1 + \ln x$

$\dfrac{dy}{dx} = x^x(1+\ln x)$

2. (8 points) Find the solution to the Initial Value Problem

$$y' = 3e^x + \sin x + 5x^2 + 17, \quad y(0) = 5.$$

$$y = 3e^x - \cos x + \frac{5x^3}{3} + 17x + C$$

$$5 = 3e^0 - \cos 0 + 0 + 0 + C$$

$$5 = 3 - 1 + C$$

$$3 = C$$

$$y = 3e^x - \cos x + \frac{5x^3}{3} + 17x + 3$$

3. (8 points) Consider a function f that has the following values and whose derivative has the following values:

x	$f(x)$	$f'(x)$
$-\pi/3$	1	-1
$-\pi/4$	2	-2
$-\pi/6$	3	-3
0	4	-4
$\pi/6$	5	-5
$\pi/4$	6	-6
$\pi/3$	7	-7

Let $k(x) = f(\arctan x)$.

(a) Write an expression for $k'(x)$. use chain rule

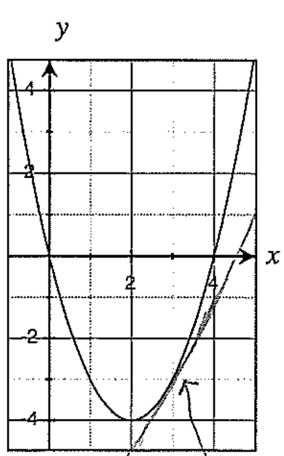
$$k'(x) = f'(\arctan x) \cdot \frac{1}{1+x^2}$$

(b) Find $k'(1)$.

$$k'(1) = f'(\arctan 1) \cdot \frac{1}{1+1}$$

$$= f'(\pi/4) \cdot \frac{1}{2} = -6 \cdot \frac{1}{2} = -3$$

4. (8 points) Consider a function g that has the following graph.



Let $h(x) = x^2 g(x)$. use product rule.

(a) Write an expression for $h'(x)$.

$$h'(x) = 2x g(x) + x^2 g'(x)$$

(b) Estimate $h'(3)$.

$$h'(3) = 2 \cdot 3\, g(3) + 3^2 g'(3)$$

$$= 6(-3) + 9(2) = 0$$

slope of this line $= g'(3) = 2$

5. (a) (8 points) Use the fact that $\tan t = \dfrac{\sin t}{\cos t}$ to show that $\dfrac{d}{dt}[\tan t] = \sec^2 t$.

$$\frac{d}{dt}(\tan t) = \frac{d}{dt}\left(\frac{\sin t}{\cos t}\right) = \frac{(\cos t)(\cos t) - (\sin t)(-\sin t)}{\cos^2 t}$$

use quotient rule

$$= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{1}{\cos^2 t} = \sec^2 t$$

(b) (7 points) Compute the following limit (if you use L'Hopital's rule be sure to show that this is a case where you can use the rule).

$$\lim_{t\to 0} \frac{\tan t}{t}$$

$$\frac{\lim_{t\to 0} \tan t}{\lim_{t\to 0} t} = \frac{0}{0}$$

this is a form where we can use L'Hopital's Rule.

$$\lim_{t\to 0} \frac{\tan t}{t} = \lim_{t\to 0} \frac{(\tan t)'}{(t)'} = \lim_{t\to 0} \frac{\sec^2 t}{1} = \sec^2 0 = 1$$

6. (7 points) Compute the following limit (if you use L'Hopital's rule be sure to show that this is a case where you can use the rule).

$$\lim_{x\to\infty} \frac{(3x^4 + 7x^2 - 8x)\cdot 1/x^4}{(12x^4 + 3x^3 - 2)\cdot 1/x^4}$$

$$= \lim_{x\to\infty} \frac{3 + 7/x^2 - 8/x^3}{12 + 3/x - 2/x^4} = \frac{3+0-0}{12+0-0} = \frac{1}{4}$$

7. Consider the function $f(x) = 5x^{2/3} - x^{5/3}$, with first derivative given below:

$$f'(x) = \frac{10 - 5x}{3\sqrt[3]{x}}$$

(a) (6 points) Find the equation of the line tangent to f at the point $(8, -12)$.

$$\text{slope} = f'(8) = \frac{10 - 5(8)}{3\sqrt[3]{8}} = \frac{10-40}{3 \cdot 2} = -5$$

$$y - -12 = -5(x-8)$$

$$\boxed{y + 12 = -5(x-8)}$$

(b) (8 points) Find the critical points of f. (Hint: there are two of them.) Classify each as a local maximum, local minimum, or neither.

$$f'(x) = 0 = \frac{10-5x}{3\sqrt[3]{x}} \Rightarrow 10 - 5x = 0$$
$$x = 2$$

$$f'(x) \text{ undefined} \Rightarrow 3\sqrt[3]{x} = 0$$
$$x = 0$$

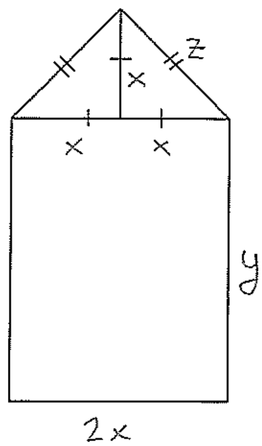
	-1	0	1	2	3
$f'(x)$	$-$		$+$		$-$
$f(x)$	$\searrow$		$\nearrow$		$\searrow$

$\Rightarrow$ so $x=0$ is a local min
$x=2$ is a local max

8. (1 point) What is your favorite function?

mine is $f(x) = e^x$, but there's no wrong answer except to leave this blank.

9. (15 points) A window has the shape of a rectangle surmounted by an isosceles triangle. (Equal sides are indicated by hash marks.) If the area of the window is 100 square feet, find the dimensions of the rectangle that will produce the smallest perimeter for the window. Neglect the thickness of the frame. (Helpful formula related to triangles: Area $= \frac{1}{2}bh$)



(a) What quantity are you trying to optimize? Are you trying to minimize it or maximize it?

perimeter of window

minimize

(b) What is the objective function for the quantity you are trying to optimize?

$$\boxed{P = 2y + 2x + 2z}$$

$$z^2 = x^2 + x^2$$

$$z = \sqrt{2x^2} = \sqrt{2}x$$

(c) Find the constraint equation and use it to rewrite the objective function from (b) as a function of one variable.

$$A = 100 = 2xy + \tfrac{1}{2}(2x)(x)$$

$$\boxed{100 = 2xy + x^2}$$

$$2xy = 100 - x^2$$

$$y = \frac{100 - x^2}{2x}$$

$$P = 2\left(\frac{100 - x^2}{2x}\right) + 2x + 2\sqrt{2}x$$

$$P = \frac{100}{x} + x + 2\sqrt{2}x$$

(d) Find the critical point(s) of the objective function. Verify that you have the desired max or min. (You may assume that the max or min happens at the critical point(s).)

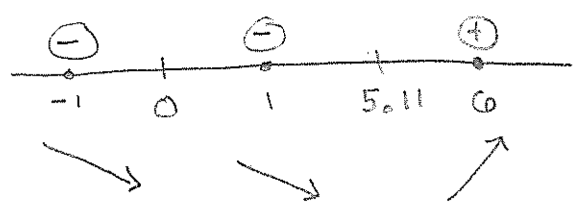
$$P' = \frac{-100}{x^2} + 1 + 2\sqrt{2} = 0$$

$$1 + 2\sqrt{2} = \frac{100}{x^2}$$

$$x^2 = \frac{100}{1 + 2\sqrt{2}}$$

$$x = 5.11$$

also P undefined when $x = 0$



so 5.11 is a min

(e) What are the dimensions of the rectangle?

$$2x = 10.22$$

$$y = \frac{100 - 5.11^2}{2(5.11)} = 7.23$$

$\boxed{\text{10.22 feet by 7.23 feet}}$