

Math 105: Review for Exam II - Solutions

1. Find dy/dx for each of the following.

(a) $y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + (\ln 2)x + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2 + \ln 2 \quad \text{Note that } e^2, \ln 2, \text{ and } \arctan 2 \text{ are constants.}$$

(b) $y = \sqrt{x} \cdot \arctan(5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}$$

(c) $y = \ln(\tan(2^{\cos(x^2)}))$

$$\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})} \cdot \sec^2(2^{\cos(x^2)}) \cdot \ln 2(2^{\cos(x^2)}) \cdot (-\sin(x^2)) \cdot 2x$$

(d) $y = \frac{x + e^\pi}{\cos 4 + \sin^5(6x)}$

Note that e^π and $\cos 4$ are constants.

$$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^\pi)(5 \sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2} \quad \text{Recall that } \sin^5(6x) = (\sin(6x))^5.$$

(e) $y = (x^2 + 1)^{\sin x}$

We need logarithmic differentiation here.

$$\ln y = \sin x \cdot \ln(x^2 + 1)$$

Take natural log of each side.

$$\frac{1}{y} \frac{dy}{dx} = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x$$

Differentiate.

$$\frac{dy}{dx} = \left[\cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] y$$

Solve for $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \left[\cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot (x^2 + 1)^{\sin x}$$

Replace y .

2. Consider the curve defined by $x^3 + y^3 = \frac{9}{2}xy$ (known as the Folium of Descartes).

(a) Find dy/dx .

Use implicit differentiation.

$$\begin{aligned} 3x^2 + 3y^2 \frac{dy}{dx} &= \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} \left(3y^2 - \frac{9}{2}x \right) &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} &= \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \end{aligned}$$

(b) Verify that the point (1,2) is on the curve above.

We must check to see if the values $x = 1$ and $y = 2$ satisfy the equation above.

$$\begin{aligned} x^3 + y^3 &\stackrel{?}{=} \frac{9}{2}xy \\ 1^3 + 2^3 &\stackrel{?}{=} \frac{9}{2} \cdot 1 \cdot 2 \\ 9 &\stackrel{?}{=} 9 \end{aligned}$$

Thus, the point (1,2) is on the curve.

- (c) Find the equation of the tangent line at the point (1,2).

We want $y = mx + b$.

$$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$

Now plug in $x = 1$ and $y = 2$ to find b .

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have $y = \frac{4}{5}x + \frac{6}{5}$.

3. Evaluate the following limits.

Throughout this solution, the symbol \star will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form $0/0$; this may be $\stackrel{0/0}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or $\stackrel{0/0}{\sim}$ or "has the form $\frac{0}{0}$, and so, by L'Hopital's Rule, is equal to" or something else. The symbol \heartsuit will serve the same purpose for the indeterminate forms ∞/∞ and $-\infty/\infty$.

(a) $\lim_{x \rightarrow 1} \frac{x^3 - 1}{7 - 7x} \star \lim_{x \rightarrow 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}$

(b) $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3^x} = \frac{0}{1} = 0$

Can't use (and don't need) L'Hopital's Rule!

(c) $\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{5x^2} \star \lim_{x \rightarrow 0} \frac{4 \sin(4x)}{10x} \star \lim_{x \rightarrow 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5}$

(d) $\lim_{x \rightarrow \infty} \frac{x^2}{2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{\ln 2 \cdot 2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0$

(e) $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{503x}$ [Students in the 8:00 and 9:30 sections may omit this problem.]

Let $L = \lim_{x \rightarrow \infty} \left(1 + \frac{4}{x}\right)^{503x}$.

This is of the form 1^∞ , so we take the \ln of each side to turn it into a L'Hopital's Rule problem.

$$\ln L = \lim_{x \rightarrow \infty} \ln \left(1 + \frac{4}{x}\right)^{503x}$$

$$\ln L = \lim_{x \rightarrow \infty} (503x) \ln \left(1 + \frac{4}{x}\right)$$

Apply the log rule $\ln(a^b) = b \ln(a)$.

$$\ln L = \lim_{x \rightarrow \infty} \frac{\ln \left(1 + \frac{4}{x}\right)}{\frac{1}{503x}}$$

Rewrite so it takes the form $\frac{0}{0}$.

$$\ln L = \lim_{x \rightarrow \infty} \frac{\frac{1}{\left(1 + \frac{4}{x}\right)} \cdot \frac{-4}{x^2}}{\frac{-1}{503x^2}}$$

Apply L'Hopital's Rule.

$$\ln L = \lim_{x \rightarrow \infty} \frac{\frac{1}{\left(1 + \frac{4}{x}\right)} \cdot 4}{\frac{1}{503}}$$

Cancel terms.

$$\ln L = \frac{1+0}{\frac{1}{503}} \cdot 4$$

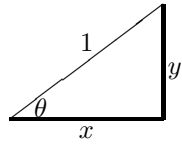
Take the limit.

$$\ln L = 4 \cdot 503$$

Now, since $\ln L = 4 \cdot 503 = 2012$, we have $L = e^{2012}$.

4. Rewrite $\tan(\arccos x)$ as an algebraic expression - no trigonometric or inverse trigonometric functions. [Students in the 8:00, 9:30, and 1:10 sections may omit this problem.]

Let $\theta = \arccos x$. That is, θ is the angle whose cosine is x .



$$x^2 + y^2 = 1^2 \Rightarrow y = \sqrt{1 - x^2}$$

$$\tan(\arccos x) = \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{y}{x} = \frac{\sqrt{1 - x^2}}{x}$$

5. Consider the function $f(x) = x^4 e^x$ with domain all real numbers.

- (a) Find the x -value(s) of all roots (x -intercepts) of f .

The equation $x^4 e^x = 0$ means $x^4 = 0$ (that is, $x = 0$) or $e^x = 0$ (no solution), so the only root is at $x = 0$.

- (b) Find the x - and y -value(s) of all critical points and identify each as a local max, local min, or neither.

$$\begin{aligned} f'(x) &= 4x^3 e^x + x^4 e^x \\ 0 &= x^3 e^x (4 + x) \\ \Rightarrow x &= 0, -4 \end{aligned}$$

Note that e^x is never 0.

	$x < -4$	$-4 < x < 0$	$4 < x$
f'	positive	negative	positive
f	\nearrow	\searrow	\nearrow

$$y\text{-values: } f(-4) = 256e^{-4} \approx 4.689, f(0) = 0$$

So, f has a local maximum at $(-4, 256e^{-4})$ and a local minimum at $(0, 0)$.

- (c) Find the x - and y -value(s) of all global extrema and identify each as a global max or global min.

There is a global minimum at $(0, 0)$. There is no global maximum because as $x \rightarrow \infty$, $f(x) \rightarrow \infty$. Note that as $x \rightarrow -\infty$, $f(x) \rightarrow 0$. You can verify this by using L'Hopital's Rule on x^4/e^{-x} .

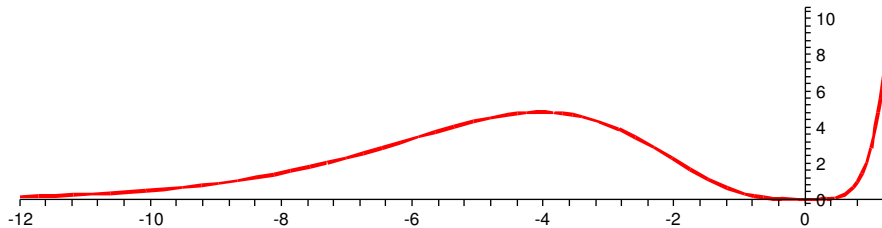
- (d) Find the x -value(s) of all inflection points.

$$\begin{aligned} f''(x) &= 12x^2 e^x + 4x^3 e^x + 4x^3 e^x + x^4 e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.} \\ 0 &= e^x (x^4 + 8x^3 + 12x^2) \\ 0 &= e^x x^2 (x^2 + 8x + 12) \\ 0 &= e^x x^2 (x + 2)(x + 6) \\ \Rightarrow x &= 0, -2, -6 \end{aligned}$$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
f''	positive	negative	positive	positive
f	concave up	concave down	concave up	concave up

So, the x -values of the inflection points of f are $x = -2$ and $x = -6$ but NOT $x = 0$.

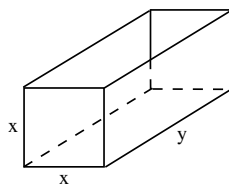
- (e) Sketch f .



6. How would your answers to the previous question change if the domain of f were $[-10, 10]$?

There would be a global maximum at $(10, 10^4 e^{10})$. (And the graph would be restricted to $-10 \leq x \leq 10$).

7. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is \$288. If the glass for the sides costs \$12 per square foot and the opaque material for the bottom costs \$3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.



Goal: Maximize volume

Objective function: volume = $V = x \cdot x \cdot y = x^2 y$

We need to get this down to a function of just one variable, so we use the *constraint equation*:

$$\text{total cost} = (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides})$$

$$288 = 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy$$

$$288 = 27xy + 24x^2$$

$$288 - 24x^2 = 27xy$$

$$\frac{288 - 24x^2}{27x} = y$$

Substituting this back into the objective function gives

$$V = x^2 y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).$$

Now that we have V as a function of just one variable, we find its maximum.

$$V'(x) = \frac{1}{27}(288 - 72x^2)$$

$$0 = \frac{1}{27}(288 - 72x^2)$$

$$0 = (288 - 72x^2)$$

$$72x^2 = 288$$

$$x^2 = \frac{288}{72}$$

$$x = 2$$

We discard $x = -2$ because lengths must be nonnegative.

Since V' is positive for $x < 2$ and negative for $2 < x$, we know that the maximum occurs at $x = 2$.

And $y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}$, so the dimensions are 2 by 2 by $\frac{32}{9}$.