**Mid-term Exam #2**
**MATH 106-D, Fall 2015**

**Name:** ______________________________________________

**Instructions:**

- Please answer as many of the following questions as possible.
- No cell phones or collaboration allowed. If you leave the classroom during the exam you must leave your cell phone with the instructor.
- Approved calculators are allowed.
- Additional scrap paper is available upon request.
- *Multiple choice questions:* Circle the letter corresponding to your answer. No partial credit will be awarded.
- *Short answer questions:* Show all of your work on the page of the problem. Clearly indicate your answer and the reasoning that you used to arrive at the answer. You do not have to simplify algebraic expressions.

This exam has 4 multiple choice problems and 4 short answer problems. There are a total of 100 points.

Good luck!

| **Problem** | **Possible Points** | **Points Earned** |
|---|---|---|
| MC | 20 | |
| 5 | 30 | |
| 6 | 20 | |
| 7 | 16 | |
| 8 | 14 | |
| **TOTAL** | **100** | |

1. (5 points) Which trigonometric substitution is needed to evaluate the integral

$$\int \frac{1}{(x^2-16)^{3/2}}\,dx?$$

   A. $x = \sec\theta$

   B. $x = 16\sin\theta$

   C. $x = 4\sin\theta$

   D. $x = 4\sec\theta$

   E. $x = \frac{1}{4}\sec\theta$

   SOLUTION: Correct answer: D.

2. (5 points) Which of the following is the correct form for the partial fraction decomposition of

$$\frac{x^2+5}{(x+1)^2(x+3)}?$$

   A. $\dfrac{Ax^2+B}{(x+1)^2} + \dfrac{Cx^2+D}{x+1} + \dfrac{Ex^2+F}{x+3}$

   B. $\dfrac{Ax+B}{(x+1)^2} + \dfrac{C}{x+1} + \dfrac{D}{x+3}$

   C. $\dfrac{A}{(x+1)^2} + \dfrac{B}{x+3}$

   D. $\dfrac{A}{(x+1)^2(x+2)} + \dfrac{B}{(x+1)(x+3)} + \dfrac{C}{x+3}$

   E. $\dfrac{A}{(x+1)^2} + \dfrac{B}{x+1} + \dfrac{C}{x+3}$

   SOLUTION: Correct answer: E.

3. (5 points) Suppose that the third-order Maclaurin polynomial for a function $f$ is given by

$$P_3(x) = 2 - \frac{x}{3} + 4x^2 - \frac{x^3}{11}.$$

What is $f''(0)$?

A. $f''(0) = 6$

B. $f''(0) = \frac{1}{3}$

C. $f''(0) = 2$

D. $f''(0) = 8$

E. $f''(0) = 0$

SOLUTION: Correct answer: D.

4. (5 points) Which of the following statements is true regarding the improper integral

$$\int_1^\infty \frac{2x}{1+x^2}\, dx?$$

A. For $x \geq 1$, $\dfrac{2x}{1+x^2} \leq 2x$, therefore the integral converges.

B. For $x \geq 1$, $\dfrac{2x}{1+x^2} \geq \dfrac{1}{x}$, therefore the integral converges.

C. For $x \geq 1$, $\dfrac{2x}{1+x^2} \geq \dfrac{1}{x}$, therefore the integral diverges.

D. $\displaystyle\int_1^\infty \frac{2x}{1+x^2}\, dx = 1.$

E. For $x \geq 1$, $\dfrac{2x}{1+x^2} \geq 2x$, therefore the integral diverges.

SOLUTION: Correct answer: C.

5. (30 points) Evaluate the following integrals using calculus. Show your work.

(a) (10 points) $\int \tan^3 x \sec^2 x \, dx$

SOLUTION: Let $u = \tan x$, then $du = \sec^2 x \, dx$.

$$\begin{aligned} \int \tan^3 x \sec^2 x \, dx &= \int u^3 \, du \\ &= \frac{u^4}{4} + C \\ &= \frac{1}{4} \tan^4 x + C \end{aligned}$$

Alternatively, set $u = \sec x$, then $du = \tan x \sec x dx$.

$$\begin{aligned} \int \tan^3 x \sec^2 x \, dx &= \int \tan^2 x \sec x (\tan x \sec x) \, dx \\ &= \int (\sec^2 x - 1) \sec x (\tan x \sec x) \, dx \\ &= \int (u^2 - 1) u \, du \\ &= \int u^3 - u \, du \\ &= \frac{u^4}{4} - \frac{u^2}{2} + C \\ &= \frac{1}{4} \sec^4 x - \frac{1}{2} \sec^2 x + C \end{aligned}$$

(b) (10 points) $\int (ax + b) \sin x \, dx$, where $a, b$ are constants.

SOLUTION: Use integration by parts,

$$\begin{aligned} u &= ax + b & dv &= \sin x \, dx \\ du &= a \, dx & v &= -\cos x \end{aligned}$$

$$\begin{aligned} \int (ax + b) \sin x \, dx &= (ax + b)(-\cos x) - \int a(-\cos x) \, dx \\ &= -(ax + b) \cos x + a \int \cos x \, dx \\ &= -(ax + b) \cos x + a \sin x + C \end{aligned}$$

(c) (10 points) $\int \frac{x+1}{x^2+x-2}\,dx$

SOLUTION: By partial fraction decomposition,

$$\frac{x+1}{x^2+x-2} = \frac{A}{x+2} + \frac{B}{x-1}.$$

Solving for $A, B$ we get $A = 3$ and $B = -2$. Then

$$\begin{aligned} \int \frac{x+1}{x^2+x-2}\,dx &= \int \frac{3}{x+2}\,dx + \int \frac{-2}{x-1}\,dx \\ &= 3\ln|x+2| - 2\ln|x-1| + C \end{aligned}$$

6. (20 points)

(a) (10 points) Evaluate the indefinite integral $\int \sqrt{4-x^2}\,dx$. You may find the following trigonometric identities helpful:

$$\sin(2\alpha) = 2\sin\alpha\cos\alpha,$$
$$\cos(2\alpha) = 2\cos^2\alpha - 1.$$

SOLUTION: Let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$.

$$\begin{aligned}
\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-4\sin^2\theta}\,(2\cos\theta)\,d\theta \\
&= 4\int \sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\
&= 4\int \cos^2\theta\,d\theta \\
&= 2\int \cos(2\theta)+1\,d\theta \\
&= 2\left(\frac{1}{2}\sin(2\theta)+\theta\right)+C \\
&= \sin(2\theta)+2\theta+C \\
&= 2\sin\theta\cos\theta+2\theta+C.
\end{aligned}$$

Next build a right triangle. By the substitution $x = 2\sin\theta$, the side opposite the angle $\theta$ will have length $x$, the hypotenuse will have length 2, and by Pythagorean Theorem, the adjacent side will have length $\sqrt{4-x^2}$. Then

$$2\sin\theta\cos\theta+2\theta+C = \frac{x\sqrt{4-x^2}}{2} + 2\arcsin\left(\frac{x}{2}\right)+C.$$

(b) (10 points) Evaluate the following improper integral.

$$\int_0^\infty xe^{-x}\,dx$$

SOLUTION: To find an antiderivative of $xe^{-x}$ integrate by parts:

$$\begin{array}{ll} u = x & dv = e^{-x}\,dx \\ du = dx & v = -e^{-x}. \end{array}$$

$$\begin{aligned} \int xe^{-x}\,dx &= -xe^{-x} - \int -e^{-x}\,dx \\ &= -xe^{-x} - e^{-x}. \end{aligned}$$

Now

$$\begin{aligned} \int_0^\infty x^{e^{-x}}\,dx &= \lim_{t\to\infty} \int_0^t x^{e^{-x}}\,dx \\ &= \lim_{t\to\infty} \left( -xe^{-x} - e^{-x}\Big|_0^t \right) \\ &= \lim_{t\to\infty} \left( -te^{-t} - e^{-t} - (0-1) \right) \\ &= \lim_{t\to\infty} \left( 1 - te^{-t} - e^{-t} \right). \end{aligned}$$

By L'Hôpital's Rule, $\lim_{t\to\infty} te^{-t} = 0$. Also we know that $\lim_{t\to\infty} e^{-t} = 0$, therefore

$$\int_0^\infty xe^{-x}\,dx = 1.$$

7. (16 points) This problem concerns the function $f(x) = \ln(x+1)$.

(a) (8 points) Find the fourth-order Taylor polynomial $P_4(x)$ for $f(x)$ centered at $x_0 = 1$. *You do not have to simplify coefficients.*

SOLUTION:

$$\begin{array}{ll} f(x) = \ln(x+1) & f(1) = \ln(2) \\ f'(x) = \frac{1}{x+1} & f'(1) = \frac{1}{2} \\ f''(x) = -\frac{1}{(x+1)^2} & f''(1) = -\frac{1}{4} \\ f^{(3)}(x) = \frac{2}{(x+1)^3} & f^{(3)}(1) = \frac{1}{4} \\ f^{(4)}(x) = -\frac{6}{(x+1)^4} & f^{(4)}(1) = -\frac{3}{8} \end{array}$$

$$\begin{aligned} P_4(x) &= \ln(2) + \frac{1}{2}(x-1) - \frac{1}{4}\frac{(x-1)^2}{2} + \frac{1}{4}\frac{(x-1)^3}{3!} - \frac{3}{8}\frac{(x-1)^4}{4!} \\ &= \ln(2) + \frac{x-1}{2} - \frac{(x-1)^2}{8} + \frac{(x-1)^3}{24} - \frac{(x-1)^4}{64} \end{aligned}$$

(b) (8 points) Taylor's Theorem states that $|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}$ for all values of $x$ in an interval $I$ containing $x_0$. What is the maximum error committed by using $P_4(x)$ (as in part (a)) over the interval $[\frac{1}{2}, \frac{3}{2}]$? *You do not have to simplify your answer.*

SOLUTION: $f^{(5)}(x) = \frac{24}{(x+1)^5}$. On the interval $[\frac{1}{2}, \frac{3}{2}]$, $f^{(5)}(x)$ reaches its maximum at $x = \frac{1}{2}$. Here, $f^{(5)}(\frac{1}{2}) = \frac{24}{(3/2)^5} = \frac{256}{81}$. Also when $\frac{1}{2} \le x \le \frac{3}{2}$, $|x-1| \le \frac{1}{2}$. Then by Taylor's theorem,

$$\begin{aligned} |f(x) - P_4(x)| &\le \frac{256/81}{5!}\left(\frac{1}{2}\right)^5 \\ &= \frac{32}{1215}\frac{1}{32} \\ &= \frac{1}{1215} = 0.000823 \end{aligned}$$

8. (14 points)

(a) (6 points) Use comparison to determine if the following improper integral converges or diverges. Justify your answer.

$$\int_1^\infty \frac{x^{1/2}-1}{x^3+x}\,dx$$

SOLUTION: For $x \geq 1$, $x^3 + x > x^3$, so $\frac{1}{x^3+x} < \frac{1}{x^3}$. In addition, $0 \leq x^{1/2} - 1 < x^{1/2}$. Therefore

$$0 \leq \frac{x^{1/2}-1}{x^3-x} < \frac{x^{1/2}}{x^3} = \frac{1}{x^{5/2}}.$$

The integral $\int_1^\infty \frac{dx}{x^{5/2}}$ converges by the $p$-test (here $p = \frac{5}{2} > 1$), thus by the comparison test, $\int_1^\infty \frac{x^{1/2}-1}{x^3-x}\,dx$ converges.

(b) (8 points) Consider the function

$$f(x) = \begin{cases} \dfrac{k}{\sqrt{x-1}} & \text{for } 1 < x < 10, \\ 0 & \text{otherwise.} \end{cases}$$

For what value of $k$ is $f(x)$ a probability density function?

SOLUTION: Find $k$ such that $\int_{-\infty}^{\infty} f(x)\,dx = 1$, where

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_1^{10} \frac{k}{\sqrt{x-1}}\,dx.$$

This is an improper integral because the integrand is infinite at 1. So

$$\begin{aligned} \int_1^{10} \frac{k}{\sqrt{x-1}}\,dx &= k \lim_{t\to 1^+} \int_t^{10} \frac{1}{\sqrt{x-1}}\,dx \\ &= k \lim_{t\to 1^+} \left( 2\sqrt{x-1}\Big|_t^{10} \right) \\ &= k \lim_{t\to 1^+} \left( 2\sqrt{9} - 2\sqrt{t-1} \right) \\ &= k(6). \end{aligned}$$

Thus $6k = 1$, so $k = \frac{1}{6}$ will make $f(x)$ a probability density function.