Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers.

1. Evaluate the following integrals.

(a) $\int \cos^{12} x \sin^3 x \, dx$ 

Break off one factor of $\sin x$ & replace $\sin^2 x$ with $1 - \cos^2 x$

$\int \cos^{12} x \sin^3 x \, dx = \int \cos^{12} x \sin^2 x \sin x \, dx = \int \cos^{12} x (1 - \cos^2 x) \sin x \, dx$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x \, dx$

$\int \cos^{12} x \sin^3 x \, dx = \int (u^{12} - u^{13}) \, du$

$= \frac{u^{13}}{13} - \frac{u^{14}}{14} + C$

$= \frac{\cos^{13} x}{13} - \frac{\cos^{14} x}{14} + C$

(b) $\int x^2 \sin x \, dx$

Use Integration by Parts

$u = x^2 \quad du = 2x \, dx$

$dv = \sin x \, dx \quad v = -\cos x$

$\int x^2 \sin x \, dx = -x^2 \cos x + 2 \int x \cos x \, dx$

IBP again! $\quad u = x \quad dv = \cos x \, dx \quad du = dx \quad v = \sin x$

$\int x \cos x \, dx = -x \cos x + \int \sin x \, dx$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + C$
(c) \[ \int \frac{4x^2 - 7x + 1}{(x - 1)^2(x + 1)} \, dx \]  

Partial Fraction Decomposition:  

\[ \frac{4x^2 - 7x + 1}{(x-1)^2(x+1)} = A \frac{1}{x-1} + B \frac{1}{(x-1)^2} + C \frac{1}{x+1} \]

So  

\( 4x^2 - 7x + 1 = A(x-1)(x+1) + B(x+1) + D(x-1)^2 \)

\[ \begin{align*}  
  x = 1 & \Rightarrow -2 = 2B \Rightarrow B = -1 \\
  x = -1 & \Rightarrow 12 = 4D \Rightarrow D = 3 \\
  x = 0 & \Rightarrow 1 = -A - 1 + 3 \Rightarrow A = 1 
\end{align*} \]

\[ \begin{align*}  
  \int \frac{4x^2 - 7x + 1}{(x-1)^2(x+1)} \, dx &= \left\{ \frac{1}{x-1} - \frac{1}{(x-1)^2} + \frac{3}{x+1} \right\} \, dx \\
  &= \int \frac{1}{x-1} \, dx - \int (x-1)^{-2} \, dx + 3 \int \frac{1}{x+1} \, dx \\
  &\text{let } u = x - 1 \Rightarrow du = dx \\
  &= \ln |u| + \frac{1}{u} + 3 \ln |u| + C \\
  &= \ln |x-1| + \frac{1}{x-1} + 3 \ln |x+1| + C 
\end{align*} \]

(d) \[ \int \frac{\sqrt{4 - x^2}}{x^2} \, dx \]  

Use Trig Substitution. 

Let \( x = 2 \sin \theta \) (for \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\)), then \( dx = 2 \cos \theta \, d\theta \)

\[ \begin{align*}  
  \int \frac{\sqrt{4 - x^2}}{x^2} \, dx &= \int \frac{\sqrt{4 - 4 \sin^2 \theta}}{4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta \\
  &= \int \frac{\sqrt{4 \cos^2 \theta}}{4 \sin^2 \theta} \cdot 2 \cos \theta \, d\theta \\
  &= \int \frac{2 \cos \theta}{4 \sin^2 \theta} \, d\theta \\
  &= \int \frac{\cos \theta}{2 \sin^2 \theta} \, d\theta \\
  &= \left\{ \frac{1 - \sin^2 \theta}{\sin^2 \theta} \right\} \, d\theta \\
  &= \left\{ \frac{1}{\sin^2 \theta} - 1 \right\} \, d\theta = \left\{ \csc^2 \theta - 1 \right\} \, d\theta \\
  &= - \cot \theta - \theta + C \\
  &= - \frac{\sqrt{4 - x^2}}{x} - \arcsin \left( \frac{x}{2} \right) + C 
\end{align*} \]
2. The following integral is improper. Either compute its value to show it converges, or show it diverges.

\[ \int_0^3 \frac{1}{\sqrt{x} - 1} \, dx \]

Let \( u = x - 1 \)
\[ du = dx \]

Change limits of integration:
\[ x = 0 \Rightarrow u = 0 - 1 = -1 \]
\[ x = 3 \Rightarrow u = 3 - 1 = 2 \]

\[ \int_0^3 \frac{-1}{\sqrt{x} - 1} \, dx = \int_{-1}^{2} \frac{1}{\sqrt{u}} \, du \]

\[ = \int_{-1}^{0} \frac{1}{\sqrt{u}} \, du + \int_{0}^{2} \frac{1}{\sqrt{u}} \, du \]

\[ = \lim_{b \to 0^-} \int_{-1}^{0} u^{-\frac{1}{2}} \, du + \lim_{a \to 0^+} \int_{a}^{2} u^{-\frac{1}{2}} \, du \]

\[ = \lim_{b \to 0^-} \frac{2}{3} u^{\frac{3}{2}} \bigg|_{-1}^{0} + \lim_{a \to 0^+} \frac{2}{3} u^{\frac{3}{2}} \bigg|_{a}^{2} \]

\[ = \lim_{b \to 0^-} \frac{2}{3} \left( \sqrt{b} - \sqrt{1} \right) + \lim_{a \to 0^+} \frac{2}{3} \left( \sqrt{4} - \sqrt{a^2} \right) \]

\[ = \frac{3}{2} (-1) + \frac{3}{2} (\sqrt{4}) = 0.88 \]
3. Consider the function \( f(x) = \ln(1 + x) \).

(a) Find \( P_3(x) \), the 3\(^{rd} \) order Taylor Polynomial, of \( f(x) \) centered at \( x = 0 \). Simplify your answer as much as possible, in other words, fractional coefficients must be in lowest terms.

\[
\begin{align*}
    f(x) &= \ln (1+x) \\
    f'(x) &= \frac{1}{1+x} \\
    f'(0) &= 1 \\
    f''(x) &= -x^{-2} \\
    f''(0) &= 0 \\
    f'''(x) &= 2(1+x)^{-3} \\
    f'''(0) &= 2
\end{align*}
\]

\[
P_3(x) = f(0) + f'(0)(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f'''(0)}{3!}(x-0)^3
\]

\[
P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}
\]

(b) Use \( P_3(x) \) to find estimates for \( \ln(0.9) \) and \( \ln(1.3) \). Show your work.

\[
\begin{align*}
    \ln(0.9) &= \ln(1+(0.9-1)) \\
    P_3(-0.1) &= (-0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} \\
    &\approx -0.105
\end{align*}
\]

Similarly, \( \ln(1.3) = \ln(1+0.3) \approx P_3(0.3) \)

\[
P_3(0.3) = 0.3 - \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} = 0.264
\]

(c) Use Taylor’s Theorem to approximate the error of your estimate for \( \ln(0.9) \), from part (b), on the interval \([-0.5, 0.4]\). Recall that error bounds for estimates using a Taylor Polynomial \( P_n(x) \) may be determined using:

\[
|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}|x-x_0|^{n+1}
\]

\[
f'(x) = \frac{1}{1+x} \\
\Rightarrow \text{to find } K_4 \text{ we need max of } \left| \frac{1}{1+x} \right| \text{ on } [-0.5, 0.4]
\]

Note: \( \left| \frac{1}{1+x} \right| \) is decreasing on \([-0.5, 0.4]\) \Rightarrow max occurs at \( x = -0.5 \) \Rightarrow \( K_4 = \left| \frac{1}{1-0.5} \right| = 2 \)

Therefore \( |f(x) - P_3(x)| \leq \frac{2}{4!} |x-0|^4 \)

\[
\leq \frac{96}{4!} |0.1|^4 \leq \frac{96}{4!} |0.5|^4 = \frac{1}{4}
\]

[BONUS] Which of the approximations in part (b) would you expect to be more accurate? Briefly explain.

(Note: simply evaluating \( \ln(0.9) \) and \( \ln(1.3) \) on your calculator to see which is closer to the estimates in (b) will not earn you any bonus points.)

\( P_n(x) \), the \( n \)th order Taylor Polynomial of \( f(x) \) centered at \( x_0 \), provides good approximations to \( f(x) \) when \( x \) is closest to \( x_0 \). In this case \( x_0 = 0 \) since 0.1 is closer to 0 than 0.5 is. We expect \( P_3(-0.1) \) to give a better estimate of \( \ln(0.9) \) than \( P_3(0.3) \) will give for \( \ln(1.3) \).
4. Consider the initial value problem \( \frac{dy}{dx} = xe^{-y} \) with \( y(0) = 3 \). Use separation of variables to find the solution to the IVP. Be sure to use the initial condition to determine the values of any constants.

\[
\frac{dy}{dx} = xe^{-y} \quad \Rightarrow \quad e^y dy = x \, dx
\]

\[
\int e^y \, dy = \int x \, dx \quad \Rightarrow \quad e^y = \frac{x^2}{2} + c
\]

Now use \( y(0) = 3 \) to find \( c \):

\[
e^3 = 0 + c \quad \Rightarrow \quad c = e^3
\]

Thus, \( e^y = \frac{x^2}{2} + e^3 \)

Now, solve for \( y \):

\[
y = \ln \left( \frac{x^2}{2} + e^3 \right)
\]

5. Consider \( \int_0^\infty \frac{1}{e^{2x} + 2700x} \, dx \). Use a comparison to determine if this integral converges or diverges? If it converges find an upper bound for its value.

Note: \( e^{2x} + 2700x \geq e^{2x} \) for \( x > 0 \)

Thus \( \frac{1}{e^{2x} + 2700x} \leq \frac{1}{e^{2x}} = e^{-2x} \) for \( x > 0 \)

And we know from work in class: \( \int_0^\infty e^{-ax} \, dx \) converges for \( a > 0 \)

Thus \( \int_0^\infty e^{-2x} \, dx \) converges and so \( \int_0^\infty \frac{1}{e^{2x} + 2700x} \, dx \) converges by comparison (since \( \int_0^\infty \frac{1}{e^{2x} + 2700x} \, dx \leq \int_0^\infty \frac{1}{e^{2x}} \, dx \))

\[
\int_0^\infty \frac{1}{e^{2x} + 2700x} \, dx \leq \int_0^\infty e^{-2x} \, dx
\]

\[
= \lim_{b \to \infty} \int_0^b e^{-2x} \, dx
\]

\[
= \lim_{b \to \infty} \left[ -\frac{1}{2} e^{-2x} \right]_0^b
\]

\[
= \lim_{b \to \infty} \left( -\frac{1}{2e^2} + \frac{1}{2e^0} \right)
\]

\[
= 0 + \frac{1}{2}
\]

Therefore, \( \int_0^\infty \frac{1}{e^{2x} + 2700x} \, dx \leq \frac{1}{2} \)
6. The magnitude of an earthquake, as measured on the Richter scale, recorded in a region of North America is a random variable $X$ with probability density given by

$$f(x) = \begin{cases} 
0.417e^{-0.417x} & \text{if } x > 0 \\
0 & \text{otherwise}
\end{cases}$$

An earthquake struck Maine on October 16, 2012 measuring 4.6 on the Richter scale. Find the probability that an earthquake striking North America will exceed a magnitude of 4.6 on the Richter scale.

Want $P(X > 4.6) = \int_{4.6}^{\infty} f(x) \, dx$

$$= \int_{4.6}^{\infty} 0.417e^{-0.417x} \, dx$$

$$= \lim_{b \to \infty} \int_{4.6}^{b} 0.417e^{-0.417x} \, dx$$

$$= \lim_{b \to \infty} \left[ e^{-0.417x} \right]_{4.6}^{b}$$

$$= \lim_{b \to \infty} \left( e^{-0.417b} - e^{-0.417 \cdot 4.6} \right)$$

$$\approx 0.147$$

About 14.7% of earthquakes in North America have magnitude 4.6 or greater.