MATH106B,C CALCULUS II - PROF. P. WONG

EXAM II - NOVEMBER 1, 2013

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	20	
5.	20	
Total	100	

1. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int e^{\sec x} \sec x \tan x \; dx.$$

We use simple substitution. Let $u = \sec x$**. Then** $du = \sec x \tan x \; dx$**. It follows that**

$$\int e^{\sec x} \sec x \tan x \; dx = \int e^u \; du = e^u + C = e^{\sec x} + C.$$

(10 pts.)(b)

$$\int \sqrt{x} \ln x \; dx.$$

We use integration by parts. Let $u = \ln x$ **and** $dv = \sqrt{x} \; dx$**. Thus,** $du = \frac{1}{x} \; dx$ **and** $v = \frac{2}{3}x^{3/2}$**. Now,**

$$\begin{aligned}\int \sqrt{x} \ln x \; dx &\overset{\textbf{IBP}}{=} (\ln x)\left(\frac{2}{3}x^{3/2}\right) - \int \frac{2}{3}x^{3/2} \cdot \frac{1}{x} \; dx \\ &= \frac{2}{3}x^{3/2} \ln x - \frac{2}{3}\int x^{1/2} \; dx \\ &= \frac{2}{3}x^{3/2} \ln x - \frac{2}{3} \cdot \frac{x^{3/2}}{3/2} + C \\ &= \frac{4}{9}x^{3/2}\left(\frac{3}{2}\ln x - 1\right) + C.\end{aligned}$$

2. Evaluate each of the following indefinite integrals (be sure to indicate what techniques you use).

(10 pts.)(a)

$$\int \frac{x}{\sqrt{1-x^4}}\ dx.$$

We use simple substitution followed by trigonometric substitution. Let $u = x^2$ **so that** $du = 2x\ dx$ **or** $x\ dx = \frac{du}{2}$**. It follows that**

$$\begin{aligned}\int \frac{x}{\sqrt{1-x^4}}\ dx &= \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} \\ &= \frac{1}{2}\arcsin u + C = \frac{1}{2}\arcsin(x^2) + C.\end{aligned}$$

(10 pts.)(b)

$$\int \frac{x^2+2x-1}{(x-1)(x^2+1)}\ dx.$$

We use the technique of partial fractions. First we write

$$\frac{x^2+2x-1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+(Bx+C)(x-1)}{(x-1)(x^2+1)}.$$

Thus,

$$x^2+2x-1 \equiv A(x^2+1)+(Bx+C)(x-1). \tag{1}$$

When $x=1$**,** (1) **yields** $2=2A$ **or** $A=1$**. Now,** (1) **becomes**

$$x^2+2x-1 \equiv x^2+1-Bx^2-Bx+Cx-C = (1+B)x^2+(-B+C)x+(1-C).$$

By equating like terms, it follows that $B=0$ **and** $C=2$**. Hence,**

$$\int \frac{x^2+2x-1}{(x-1)(x^2+1)}\ dx = \int \frac{dx}{x-1} + \int \frac{2\ dx}{x^2+1} = \ln|x-1| + 2\arctan x + C.$$

3. For each of the following improper integrals, evaluate if it exists. Justify your answer.
(10 pts.)(a)

$$\int_0^\infty xe^{-x}\ dx$$

We use the technique of integration by parts. Note that

$$\begin{aligned}\int_0^\infty xe^{-x}\ dx &= \lim_{b\to\infty}\int_0^b xe^{-x}\ dx \quad (\textbf{let } u = x \textbf{ and } dv = e^{-x}dx)\\ &= \lim_{b\to\infty}\left(-xe^{-x}\Big|_0^b - \int_0^b (-e^{-x}\ dx\right)\\ &= \lim_{b\to\infty}\left(-be^{-b} - e^{-x}\Big|_0^b\right)\\ &= \lim_{b\to\infty}(-be^{-b} - e^{-b} + 1) = 1.\end{aligned}$$

Thus this improper integral converges to 1. (Note that one can use L'Hôpital's rule to show that $be^{-b} \to 0$ as $b \to \infty$.)

(10 pts.)(b)

$$\int_1^2 \frac{3x}{\sqrt{x^2-1}}\ dx$$

First note that this integral is improper at $x = 1$ so

$$\int_1^2 \frac{3x}{\sqrt{x^2-1}}\ dx = \lim_{b\to 1^+}\int_b^2 \frac{3x}{\sqrt{x^2-1}}\ dx.$$

We use simple substitution. Let $u = x^2 - 1$ Then $du = 2x\ dx$ or $x\ dx = \frac{1}{2}du$. When $x = 2, u = 3$ and when $x = b, u = b^2 - 1$. It follows that

$$\begin{aligned}\int_1^2 \frac{3x}{\sqrt{x^2-1}}\ dx &= \lim_{b\to 1^+}\int_{b^2-1}^3 \frac{3\cdot\frac{1}{2}du}{\sqrt{u}}\\ &= \frac{3}{2}\lim_{b\to 1^+}\frac{u^{1/2}}{1/2}\Big|_{b^2-1}^3\\ &= \frac{3}{2}\lim_{b\to 1^+}\frac{\sqrt{3}}{1/2} - \frac{\sqrt{b^2-1}}{1/2}\\ &= 3\sqrt{3}.\end{aligned}$$

4. Let $f(x) = \sin(2x)$.

(9 pts.)(a) Find the third-order Taylor polynomial $P_3(x)$ of $f(x)$ based at $x_0 = \frac{\pi}{2}$.

Since $f(x) = \sin(2x)$**, it follows that** $f'(x) = 2\cos(2x), f''(x) = -4\sin(2x), f'''(x) = -8\cos(2x)$**. At** $x_0 = \frac{\pi}{2}$**, we have** $f(\frac{\pi}{2}) = 0, f'(\frac{\pi}{2}) = -2, f''(\frac{\pi}{2}) = 0, f'''(\frac{\pi}{2}) = 8$**. It follows that**

$$\begin{aligned} P_3(x) &= f\left(\frac{\pi}{2}\right) + f'\left(\frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + \frac{f''\left(\frac{\pi}{2}\right)}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{f'''\left(\frac{\pi}{2}\right)}{3!}\left(x - \frac{\pi}{2}\right)^3 \\ &= 0 + (-2)\left(x - \frac{\pi}{2}\right) + 0 + \frac{8}{3!}\left(x - \frac{\pi}{2}\right)^3 = (-2)\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3. \end{aligned}$$

(5 pts.)(b) Find the third-order Maclaurin polynomial $M_3(x)$ of $f(x)$.

Using the computation of the derivatives of $f(x)$ **in part (a), we have** $f(0) = 0, f'(0) = 2, f''(0) = 0, f'''(0) = -8$**. It follows that**

$$M_3(x) = 0 + 2(x) + 0 + \frac{(-8)}{3!}x^3 = 2x - \frac{4}{3}x^3.$$

(6 pts.)(c) What is the maximum error committed by using $P_3(x)$ (as in part (a)) over the interval $[0, \pi]$, according to Taylor's Theorem? [Hint: how do you obtain K_4?]

Note that $f'''(x) = -8\cos(2x)$ **so** $f^{(4)}(x) = 16\sin(2x)$**. Over the interval** $[0, \pi]$**,** $|f^{(4)(x)}| \leq 16$ **so we can choose** $K_4 = 16$**. By Taylor's theorem, the maximum error committed by** P_3 **is less than or equal to**

$$\frac{K_4}{4!} \cdot \left|x - \frac{\pi}{2}\right|^4 \leq \frac{16}{4!}\left(\frac{\pi}{2}\right)^4 = \frac{\pi^4}{24}.$$

5. (12 pts.)(a) Use comparison to determine whether the following improper integral converges or diverges. Justify your answer.

$$\int_2^\infty \frac{dx}{2x^3 - 3x + 2}$$

First note that when x is sufficiently large, $\frac{1}{2x^3-3x+2}$ behaves like $\frac{1}{2x^3}$ so we suspect that the improper integral in question should converge. Now, we need to show that this improper integral is smaller than another improper integral that converges. To do that, note that for $x \geq 2$,

$$2x^3 - 3x + 2 > 2x^3 - 3x = x^3 + (x^3 - 3x) > x^3.$$

so that $\frac{1}{2x^3-3x+2} < \frac{1}{x^3}$ and hence

$$\int_2^\infty \frac{dx}{2x^3 - 3x + 2} < \int_2^\infty \frac{dx}{x^3} < \int_1^\infty \frac{dx}{x^3}$$ **which converges by the p-test with $p = 3 > 1$.**

We now conclude that the improper integral $\int_2^\infty \frac{dx}{2x^3-3x+2}$ converges.

(8 pts.)(b) Consider the following function

$$f(x) = \begin{cases} 0, & \text{for } x < 0; \\ kx^2, & \text{for } 0 \leq x \leq 1; \\ 0, & \text{for } x > 1. \end{cases}$$

For what value of k is $f(x)$ a *probability density function*?

For $f(x)$ to be a p.d.f., $f(x) \geq 0$ for all x and $\int_{-\infty}^\infty f(x)\ dx = 1$. The latter condition becomes

$$\begin{aligned} 1 = \int_{-\infty}^\infty f(x)\ dx &= \int_{-\infty}^0 f(x)\ dx + \int_0^1 f(x)\ dx + \int_1^\infty f(x)\ dx \\ &= \int_0^1 kx^2\ dx = \left.\frac{kx^3}{3}\right|_0^1 \\ &= \frac{k}{3}. \end{aligned}$$

Thus, for $f(x)$ to be a p.d.f., $\frac{k}{3} = 1$ or $k = 3$.