# Math 106 Fall 2013

## Test 2 (50 points)

Name: Solutions

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are eight questions. Questions are printed on both sides of a page.

You may use any of the following facts:

$$P_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!}\, |x - x_0|^{n+1}$$

$$\int u\, dv = uv - \int v\, du$$

$\displaystyle\int_1^\infty \frac{1}{x^p}\, dx$ converges for $p > 1$ and diverges for $p \leq 1$.

$\displaystyle\int_0^1 \frac{1}{x^p}\, dx$ converges for $p < 1$ and diverges for $p \geq 1$.

$\displaystyle\int_0^\infty e^{-ax}\, dx$ converges for $a > 0$.

1. (7 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{2x^2-30x-4}{(x-5)(1+x^2)}\,dx$$

Partial fractions.

$$\frac{2x^2-30x-4}{(x-5)(1+x^2)} = \frac{A}{x-5} + \frac{Bx+C}{1+x^2}.$$

$$2x^2-30x-4 = A(1+x^2) + (Bx+C)(x-5)$$

Find $A, B, C$:

$\underline{x=5:}$ $-104 = 26A$. So $\boxed{A=-4}$

$\underline{x=0:}$ $-4 = \overset{-4}{\cancel{26}}(1) + (0+C)(-5)$. $\overset{0}{\cancel{-30}} = -5C$ $\boxed{C=0}$.

$\underline{x=1:}$ $-32 = \overset{-4}{\cancel{26}}(2) + (B+0)(-4)$ $-24 = -4B$ $\boxed{B=6}$

So $$\int \frac{2x^2-30x-4}{(x-5)(1+x^2)}\,dx = \int\left(\frac{-4}{x-5} + \frac{6x}{1+x^2}\right)dx$$

$$= \int \frac{-4}{x-5}\,dx + \int \frac{6x}{1+x^2}\,dx$$

$u = x-5$ $\quad$ $w = 1+x^2$

$du = dx$ $\quad$ $dw = 2x\,dx$. $\quad$ $dx = \frac{dw}{2x}$.

$$= \int \frac{-4}{u}\,du + 6\int \frac{x}{w}\cdot\frac{dw}{2x}$$

$$= -4\ln|u| + 3\ln|w| + C$$

$$= -4\ln|x-5| + 3\ln|1+x^2| + C.$$

2. (4 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \tan^7 x \sec^4 x \, dx$$

$u = \tan x$, $du = \sec^2 x \, dx$.

$\int \tan^7 x \sec^4 x \, dx = \int u^7 \sec^2 x \cdot \underbrace{\sec^2 x \, dx}_{du} = \int u^7 (\tan^2 x + 1) \, du$

$= \int u^7 (u^2+1) \, du = \int (u^9 + u^7) \, du$

$= \dfrac{u^{10}}{10} + \dfrac{u^8}{8} + C$

$= \dfrac{(\tan x)^{10}}{10} + \dfrac{(\tan x)^8}{8} + C$.

3. (6 points) Evaluate the following definite integral exactly. In case of an improper integral, determine the convergence of the integral. Show clearly any limit computation you do. If the integral converges, find its value. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int_{0.5}^{2} \frac{dx}{2x-1}$$

Improper at 0.5. (As $x \to 0.5^+$, $\dfrac{1}{2x-1} \to \infty$).

$\displaystyle\int_{0.5}^{2} \frac{1}{2x-1} dx = \lim_{t \to 0.5^+} \int_t^2 \frac{1}{2x-1} dx$.

$\displaystyle\int \frac{1}{2x-1} dx$ $\quad u = 2x-1$, $du = 2dx$. So $dx = \frac{1}{2} du$

$= \displaystyle\int \frac{1}{u} \frac{1}{2} du = \frac{1}{2} \ln|u| = \frac{1}{2} \ln|2x-1|$

$\displaystyle\int_t^2 \frac{1}{2x-1} dx = \frac{1}{2} \ln|2x-1| \Big]_t^2 = \frac{1}{2} \ln 3 - \frac{1}{2} \ln|2t-1|$.

$\displaystyle\lim_{t \to 0.5^+} \int_t^2 \frac{1}{2x-1} dx = \lim_{t \to 0.5^+} \left( \frac{1}{2} \ln 3 - \frac{1}{2} \ln|2t-1| \right)$

$= \displaystyle\frac{1}{2} \ln 3 - \frac{1}{2} \lim_{t \to 0.5^+} \ln|2t-1|$.

As $t \to 0.5^+$, $2t-1 \to 0^+$.
So $\ln|2t-1| \to -\infty$.
(or see graph of $\ln|2t-1|$).

$= \dfrac{1}{2} \ln 3 - (-\infty)$.

$= \infty$.

So $\displaystyle\int_{0.5}^{2} \frac{1}{2x-1} dx$ diverges.

4. (7 points) Evaluate the following integral exactly. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int \sqrt{16 - x^2}\, dx$$

$x = 4\sin t \quad dx = 4\cos t\, dt$

$\int \sqrt{16-x^2}\,dx = \int \sqrt{16 - 16\sin^2 t} \cdot 4\cos t\, dt$

$= \int \sqrt{16(1-\sin^2 t)}\ 4\cos t\, dt$

$= \int \sqrt{16\cos^2 t}\ 4\cos t\, dt$

$= \int 4\cos t \cdot 4\cos t\, dt = \int 16\cos^2 t\, dt$

Formula # 41, $a = 1$

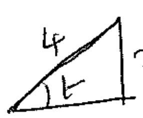
$= 16\left[\frac{t}{2} + \frac{\sin(2t)}{4}\right]$

$= 8t + 4\sin(2t)$.

$= 8t + 4(2\sin t\cos t)$

$= 8t + 8\sin t\cos t$.

$x = 4\sin t \quad \sin t = \frac{x}{4}$. So $t = \arcsin\left(\frac{x}{4}\right)$.




$\cos t = \frac{\sqrt{16-x^2}}{4}$.

$\sqrt{16-x^2}$ ← using Pythagorean theorem.

$\int \sqrt{16-x^2}\, dx = 8\arcsin\left(\frac{x}{4}\right) + 8 \cdot \frac{x}{4} \cdot \frac{\sqrt{16-x^2}}{4} + C$

$= 8\arcsin\left(\frac{x}{4}\right) + \frac{x\sqrt{16-x^2}}{2} + C$.

5. (7 points) Evaluate the following integral exactly. (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

$$\int 4x \ln x \, dx$$

Integration by parts.

$u = \ln x$. $\quad dv = 4x \, dx$.

$du = \frac{1}{x} dx$. $\quad v = \int 4x \, dx = 2x^2$.

$$\int 4x \ln x \, dx = (\ln x)(2x^2) - \int 2x^2 \cdot \frac{1}{x} dx$$

$$= 2x^2 \ln x - \int 2x \, dx$$

$$= 2x^2 \ln x - x^2 + C$$

6. (5 points) Use comparisons to determine the convergence of the following integral.

$$\int_2^\infty \frac{\sin x + 3}{5\sqrt{x^3}}\, dx\ .$$

$-1 \le \sin x \le 1$. for $x \geqslant 2$.

So $2 \le \sin x + 3 \le 4$

$$\frac{2}{5x^{3/2}} \le \underbrace{\frac{\sin x + 3}{5x^{3/2}} \le \frac{4}{5x^{3/2}}}_{\text{This comparison is useful here}} .$$

because $\int_2^\infty \frac{4}{5x^{3/2}} dx$ converges $(p = \frac{3}{2} > 1)$.

So $\int_2^\infty \frac{\sin x + 3}{5x^{3/2}} dx$ converges.

7. (6 points) The p.d.f. of a continuous random variable $X$ is given by $f(x) = \frac{k}{(x+3)^2}$ for $x \geq 1$ (the function is zero for all other values of $x$). Find $k$.

We know $\int_1^\infty f(x)\,dx = 1$.

First we find $\int_1^\infty f(x)\,dx$.

$$\int_1^\infty f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{k}{(x+3)^2}\,dx.$$

$u = x+3, \quad du = dx.$

$$\int \frac{k}{(x+3)^2}\,dx = k\int \frac{1}{u^2}\,du = k\left(\frac{-1}{u}\right) = \frac{-k}{(x+3)}.$$

$$\lim_{t\to\infty} \int_1^t \frac{k}{(x+3)^2}\,dx = \lim_{t\to\infty} \left[\frac{-k}{(x+3)}\right]_1^t$$

$$= \lim_{t\to\infty} \left[\frac{-k}{t+3} + \frac{k}{4}\right].$$

As $t \to \infty$, $t+3 \to \infty$. So $\frac{1}{t+3} \to 0$. and hence $\frac{-k}{t+3} \to 0$.

$$\text{So } \lim_{t\to\infty} \int_1^t \frac{k}{(x+3)^2}\,dx = 0 + \frac{k}{4} = \frac{k}{4}.$$

Since $\int_1^\infty f(x)\,dx = 1$, $\cancel{0+}\ \frac{k}{4} = 1$.

So $k = 4$.

8. (8 points) Suppose the fifth-order Taylor polynomial for a function $f$ based at $x_0 = 2$ is

$$P_5(x) = 9(x-2) - (x-2)^2 + \frac{1}{10}(x-2)^4 - \frac{1}{40}(x-2)^5.$$

(a) Find $f^{(5)}(2)$.

Coefficient of $(x-2)^5$ in the given polynomial is $\frac{-1}{40}$.

By the Taylor polynomial formula, coefficient of $(x-2)^5$ is $\frac{f^{(5)}(2)}{5!}$

So $\frac{f^{(5)}(2)}{5!} = -\frac{1}{40}$. So $f^{(5)}(2) = -\frac{5!}{40} = -3$.

(b) Write the third-order Taylor polynomial for $f$ based at $x_0 = 2$.

$P_3(x) = 9(x-2) - (x-2)^2$ (as seen from $P_5(x)$.)

(c) What does Taylor's theorem imply about the maximum approximation error committed by $P_5$ over the interval $[1, 4]$? Assume that $|f^{(6)}(x)| \le \sqrt{x}$ for all $x$ in $[1, 4]$.

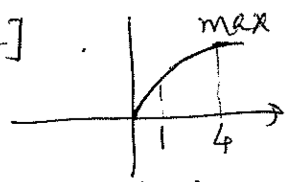
By Taylor's theorem,

$|f(x) - P_5(x)| \le \frac{K_6}{6!}|x-2|^6$ for all $x$ in $[1,4]$.

To find $K_6$:

$K_6 =$ max value of $|f^{(6)}(x)|$ in $[1,4]$.

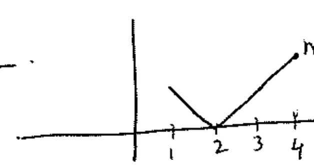
Since $|f^{(6)}(x)| \le \sqrt{x}$ for all $x$ in $[1,4]$, we find max value of $\sqrt{x}$ in $[1,4]$.




Graph of $\sqrt{x}$.

$K_6 = \sqrt{4} = 2$.

In $[1,4]$, max. value of $|x-2|$ is 2.




Graph of $|x-2|$.

So $|f(x) - P_5(x)| \le \frac{2}{6!} \cdot 2^6 = \frac{\cancel{2} \cdot 2^{\cancel{6}\,4}}{6 \times 5 \times \cancel{4} \times 3 \times \cancel{2}} = \frac{8}{45}$.

So the max. approximation error $\le \frac{8}{45}$ over the interval $[1,4]$.