1. The following questions have to do with the integral $\int \frac{\ln(x)}{x^2}\,\mathrm{d}x$

   (a) Evaluate $\int \frac{\ln(x)}{x^2}\,\mathrm{d}x$.
   Use integration by parts

$$\begin{aligned}\int \frac{1}{x^2}\cdot\ln(x)\,\mathrm{d}x = \int \left(x^{-2}\right)\cdot\ln(x)\,\mathrm{d}x = \int \left(-x^{-1}\right)'\cdot\ln(x)\,\mathrm{d}x &= -x^{-1}\ln(x) - \int -x^{-1}\cdot(\ln(x))'\,\mathrm{d}x \\ &= -x^{-1}\ln(x) + \int x^{-1}\cdot\frac{1}{x}\,\mathrm{d}x \\ &= -x^{-1}\ln(x) + \int x^{-2}\,\mathrm{d}x \\ &= -x^{-1}\ln(x) - x^{-2} + C\end{aligned}$$

You can also do parts using u and $\mathrm{d}v$:

$u = \ln(x)$	$\mathrm{d}v = x^{-2}$
$\mathrm{d}u = \frac{1}{x}$	v$= -x^{-1}$

to get

$$\int x^{-2}\cdot\ln(x)\,\mathrm{d}x = \ln(x)\cdot\left(-x^{-1}\right) - \int -x^{-1}\cdot\frac{1}{x}\,\mathrm{d}x = -x^{-1}\ln(x) - x^{-1} + C$$

(b) Suppose your answer from part (a) is $\frac{\ln x - 2x}{12x^2}$. Explain how to use this answer to understand if the integral $\int_1^\infty \frac{\ln(x)}{x^2}\, dx$ is convergent or divergent.

Assuming the answer from (a) is

$$\frac{\ln x - 2x}{12x^2},$$

the Fundamental Theorem of Calculus says

$$\int_1^t \frac{\ln(x)}{x^3}\, dx = \left(\frac{\ln x - 2x}{12x^2}\right)\Bigg|_1^t = \frac{\ln t - 2t}{12t^2} + \frac{1}{6}$$

To determine $\int_1^\infty \frac{\ln(x)}{x^2}\, dx$ we would then take the limit of $\frac{\ln t - 2t}{12t^2}$ as t goes to infinity. This limit has indeterminate form $\frac{\infty}{\infty}$ and requires L'Hospital's Rule:

$$\lim_{t\to\infty} \frac{\ln t - 2t}{12t^2} = \lim_{t\to\infty} \frac{\left(\frac{1}{t} - 2\right)}{24t} = \lim_{t\to\infty} \frac{1}{24t^2} - \frac{2}{24t} = 0 - 0 = 0.$$

Since the limit produces a real number, the improper integral $\int_1^\infty \frac{\ln(x)}{x^3}\, dx$ is convergent.

(c) Now suppose that you suspect that $\frac{7}{x^{4/3}} \leq \frac{\ln(x)}{x^3} \leq \frac{9}{x^{3/2}}$ over some interval $[a, \infty)$.

Use this idea along with the Comparison Test to determine if $\int_a^\infty \frac{\ln(x)}{x^3}\, dx$ is convergent or divergent. Remember, your answer <u>must</u> include sentences!

Because $\frac{\ln(x)}{x^2} \leq \frac{9}{x^{3/2}}$ and $\int_a^\infty \frac{9}{x^{3/2}}\, dx$ is convergent by the P-Test (with $p = 3/2 > 1$) we know that $\int_a^\infty \frac{\ln(x)}{x^3}\, dx$ is convergent by the Comparison Test.

2. Consider the following integral. Consider the following integral. $\displaystyle\int \frac{1}{4\sqrt{x+1}+x+4}\,\mathrm{d}x$

(a) Use a rationalizing substitution and related steps to convert the integral to a Case I or Case II Partial Fractions result. Your final answer for this part should be an integral of a rational function. **DO NOT SOLVE FOR COEFFICIENTS AND DO NOT INTEGRATE!**

Let $u = \sqrt{x+1}$. Then $u^2 = x + 1$ and $2u\ \mathrm{d}u = \mathrm{d}x$ so

$$\begin{aligned}\int \frac{1}{4\sqrt{x+1}+x+4}\,\mathrm{d}x &= \int \frac{2u}{4u+(u^2-1)+4}\,\mathrm{d}u\\ &= \int \frac{2u}{u^2+4u+3}\,\mathrm{d}u\\ &= \int \frac{2u}{(u+3)(u+1)}\,\mathrm{d}u\end{aligned}$$

(b) Suppose your result from part (a) is $\displaystyle\int \frac{8u}{(u-1)^2(u+1)}\,\mathrm{d}u$. Evaluate this new integral.

Write $\displaystyle\frac{8u}{(u-1)^2(u+1)} = \frac{A}{u-1} + \frac{B}{(u-1)^2} + \frac{C}{u+1}$. Then clearing fractions gives

$$8u = A(u+1)(u-1) + B(u+1) + C(u-1)^2.$$

Selecting values for u now allows us to solve for A, B, and C:

$$\begin{aligned}u = 1: &\qquad \Rightarrow 8 = B(1+1) \Rightarrow B = 4\\ u = -1: &\qquad \Rightarrow -8 = C(-1-1)^2 \Rightarrow C = -2\end{aligned}$$

Now select any other value of u and use the values of B and C to find A:

$$u = 0: \qquad 0 = A(0+1)(0-1) + 4 - 2 \Rightarrow A = 2$$

So

$$\frac{8u}{(u-1)^2(u+1)} = \frac{2}{u-1} + \frac{4}{(u-1)^2} - \frac{2}{u+1}$$

and

$$\int \frac{8u}{(u-1)^2(u+1)}\,\mathrm{d}u = \int \frac{2}{u-1}\,\mathrm{d}u + \int \frac{4}{(u-1)^2}\,\mathrm{d}u - \int \frac{2}{u+1}\,\mathrm{d}u$$

$$= 2\ln|u-1| - 4(u-1)^{-1} - 2\ln|u+1| + C$$

3. Consider the following integral. $$\int \sin^3(x)\cos^4(x)\ \mathrm{d}x$$

(a) Evaluate $\int \sin^3(x)\cos^4(x)\ \mathrm{d}x$

$$\int \sin^3(x)\cos^4(x)\ \mathrm{d}x = \int \sin^2(x)\cos^4(x)\sin(x)\ \mathrm{d}x$$

$$= \int [1-\cos^2(x)]\cos^4(x)\sin(x)\ \mathrm{d}x$$

Let $u = \cos(x) \Rightarrow \mathrm{d}u = -\sin(x)\ \mathrm{d}x$

$$\int [1-\cos^2(x)]\cos^4(x)\sin(x)\ \mathrm{d}x = \int -[1-u^2]u^4\ \mathrm{d}u = \int u^6 - u^4\ \mathrm{d}u$$

$$= \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{[\cos(x)]^7}{7} - \frac{[\cos(x)]^5}{5} + C$$

(b) The probability that Bailey falls asleep during a class within x minutes after the start is a given by the function

$$f(x) = \begin{cases} c\cdot\sin^3(x)\cos^4(x) & \text{if} \quad 0 \le x \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

where c is some constant number.

Use properties of a pdf (probability density function) and your answer from part (a) to find the value for c.

For any pdf $f(x)$ we must have

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1$$

Since our functions is zero over the intervals $(-\infty, 0)$ and (π, ∞) this means

$$1 = \int_0^{\pi} f(x)\ \mathrm{d}x = \int_0^{\pi/2} c\cdot\sin^3(x)\cos^4(x)\ \mathrm{d}x$$

According to the answer above,

$$\int_0^{\pi} \sin^3(x)\cos^4(x)\ \mathrm{d}x = \left(\frac{[\cos(x)]^7}{7} - \frac{[\cos(x)]^5}{5}\right)\Bigg|_0^{\pi} = \frac{-1}{7} - \frac{-1}{5} - \left(\frac{1}{7} - \frac{1}{5}\right) = \frac{4}{35}$$

Then

$$1 = \int_0^{\pi/2} c\cdot\cos^3(x)\sin^4(x)\ \mathrm{d}x = c\cdot\frac{4}{35}$$

implies that $c = \dfrac{35}{4} = 8.75$.

4. The following questions have to do with the integral $I = \int \sin(3x) \cdot \cos(x)\ \mathrm{d}x$.

(a) Demonstrate the results of one application of integration by parts applied to $I = \int \sin(3x) \cdot \cos(x)\ \mathrm{d}x$.

$$\begin{aligned} I = \int \sin(3x) \cdot \cos(x)\ \mathrm{d}x &= \int \sin(3x)(\sin(x))'\ \mathrm{d}x \\ &= \sin(3x)\sin(x) - \int \sin(x)(\sin(3x))'\ \mathrm{d}x \\ &= \sin(3x)\sin(x) - 3\int \sin(x)\cos(3x)\ \mathrm{d}x \end{aligned}$$

(b) Continue your work from part (a) and evaluate $I = \int \sin(3x) \cdot \cos(x)\ \mathrm{d}x$.

$$\begin{aligned} I &= \sin(3x)\sin(x) - 3\int \sin(x)\cos(3x)\ \mathrm{d}x \\ &= \sin(3x)\sin(x) - 3\int (-\cos(x))'\cos(3x)\ \mathrm{d}x \\ &= \sin(3x)\sin(x) - 3[-\cos(x)\cos(3x) - \int -\cos(x)(\cos(3x))'\ \mathrm{d}x] \\ &= \sin(3x)\sin(x) - 3[-\cos(x)\sin(3x) - 3\int \cos(x)\sin(3x)\ \mathrm{d}x] \\ &= \sin(3x)\sin(x) + 3\cos(x)\sin(3x) + 9\int \cos(x)\sin(3x)\ \mathrm{d}x] \\ &= \sin(3x)\sin(x) + 3\cos(x)\sin(3x) + 9I \end{aligned}$$

Then $I = \sin(x)\cos(3x) - 3\cos(x)\sin(3x) + 9I$ implies that $-8I = \sin(3x)\sin(x) + 3\cos(x)\sin(3x)$ or that

$$I = -\frac{1}{8}[\sin(3x)\sin(x) + 3\cos(x)\sin(3x)] + C.$$

5. Consider the limit $\lim_{x \to 0} \frac{x^2}{e^{-3x} + 3x - 1}$.

(a) Find a relevant Taylor Polynomial that will allow you to evaluate the limit. For this part, focus on calculating and assembling the pieces of your Taylor Polynomial.

We should use $a = 0$ since the limit is as $x \to 0$. We will need at least $P_2(x)$ since there is a x^2 in the denominator. Now focus on $f(x) = e^{-3x}$, the component that is not a polynomial. The appropriate derivatives and values are given in the chart below:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{-3x}	1
1	$-3e^{-3x}$	-3
2	$9e^{-3x}$	9

Assembling the pieces now gives

$$P_2(x) = f^{(0)}(0) + \frac{f^{(1)}(0)}{1!}(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 = 1 - 3x + \frac{9}{2}x^2.$$

(b) Use your Taylor Polynomial answer to evaluate the following limit:

$$\lim_{x \to 0} \frac{x^2}{e^{-3x} + 3x - 1}$$

From part (a) we see that

$$\lim_{x \to 0} \frac{x^2}{e^{-3x} + 3x - 1} = \lim_{x \to 0} \frac{x^2}{1 - 3x + \frac{9}{2}x^2 + 3x - 1} = \lim_{x \to 0} \frac{x^2}{\frac{9}{2}x^2} = \lim_{x \to 0} \frac{1}{\left(\frac{9}{2}\right)} = \frac{2}{9}.$$