1. The following questions have to do with the integral $\int \frac{\ln(x)}{x^3}\, \mathrm{d}x$

(a) Evaluate $\int \frac{\ln(x)}{x^3}\, \mathrm{d}x$.
Use integration by parts

$$\begin{aligned}
\int \frac{1}{x^3} \cdot \ln(x)\, \mathrm{d}x = \int \left(x^{-3}\right) \cdot \ln(x)\, \mathrm{d}x &= \int \left(-\frac{1}{2}x^{-2}\right)' \cdot \ln(x)\, \mathrm{d}x \\
&= -\frac{1}{2}x^{-2}\ln(x) - \int -\frac{1}{2}x^{-2} \cdot (\ln(x))'\, \mathrm{d}x \\
&= -\frac{1}{2}x^{-2}\ln(x) + \int \frac{1}{2}x^{-2} \cdot \frac{1}{x}\, \mathrm{d}x \\
&= -\frac{1}{2}x^{-2}\ln(x) + \frac{1}{2}\int x^{-3}\, \mathrm{d}x \\
&= -\frac{1}{2}x^{-2}\ln(x) - \frac{1}{4}x^{-2} + C
\end{aligned}$$

You can also do parts using u and $\mathrm{d}v$:

$u = \ln(x)$	$\mathrm{d}v = x^{-3}$
$\mathrm{d}u = \frac{1}{x}$	$\mathrm{v} = -\frac{1}{2}x^{-2}$

to get

$$\int x^{-3} \cdot \ln(x)\, \mathrm{d}x = \ln(x) \cdot \left(-\frac{1}{2}x^{-2}\right) - \int -\frac{1}{2}x^{-2} \cdot \frac{1}{x}\, \mathrm{d}x = -\frac{1}{2}x^{-2}\ln(x) - \frac{1}{4}x^{-2} + C$$

(b) Suppose your answer from part (a) is $\frac{\ln x - 4x}{12x^2}$. Explain how to use this answer to understand if the integral $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$ is convergent or divergent.

Assuming the answer from (a) is

$$\frac{\ln x - 4x}{12x^2}$$

, the Fundamental Theorem of Calculus says

$$\int_1^t \frac{\ln(x)}{x^3}\,dx = \left(\frac{\ln x - 4x}{12x^2}\right)\Bigg|_1^t = \frac{\ln t - 4t}{12t^2} - \frac{\ln(1) - 1}{3}$$

To determine $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$ we would then take the limit of $\frac{\ln t - 4t}{12t^2}$ as t goes to infinity. This limit has indeterminate form $\frac{\infty}{\infty}$ and requires L'Hospital's Rule:

$$\lim_{t\to\infty} \frac{\ln t - 4}{12t^2} = \lim_{t\to\infty} \frac{\left(\frac{1}{t} - 4\right)}{24t} = \lim_{t\to\infty} \frac{1}{24t^2} - \frac{4}{24t} = 0 - 0 = 0.$$

Since the limit produces a real number, the improper integral $\int_1^\infty \frac{\ln(x)}{x^3}\,dx$ is convergent.

(c) Now suppose that you suspect that $\frac{2}{x^{3/2}} \leq \frac{\ln(x)}{x^3} \leq \frac{5}{x^{3/2}}$ over some interval $[a, \infty)$.

Use this idea along with the Comparison Test to determine if $\int_a^\infty \frac{\ln(x)}{x^3}\,dx$ is convergent or divergent. Remember, your answer <u>must</u> include sentences!

Because $\frac{\ln(x)}{x^3} \leq \frac{5}{x^{3/2}}$ and $\int_a^\infty \frac{5}{x^{3/2}}\,dx$ is convergent by the P-Test (with $p = 3/2 > 1$) we know that $\int_a^\infty \frac{\ln(x)}{x^3}\,dx$ is convergent by the Comparison Test.

2. Consider the following integral. $$\int \frac{5}{x-\sqrt{x+6}}\,\mathrm{d}x$$

(a) Use a rationalizing substitution and related steps to convert the integral to a Case I or Case II Partial Fractions result. Your final answer for this part should be an integral of a rational function. **DO NOT SOLVE FOR COEFFICIENTS AND DO NOT INTEGRATE!**

Let $u = \sqrt{x+2}$. Then $u^2 = x+6 \Leftrightarrow x = u^2 - 6$ and $2u\ \mathrm{d}u = \mathrm{d}x$ so

$$\int \frac{5}{x-\sqrt{x+6}}\,\mathrm{d}x = \int \frac{10u}{u^2-6-u}\,\mathrm{d}u = \int \frac{10u}{u^2-u-6}\,\mathrm{d}u = \int \frac{10u}{(u-3)(u+2)}\,\mathrm{d}u$$

(b) Suppose your result from part (a) is $\int \frac{8u}{(u-1)(u+1)^2}\,\mathrm{d}u$. Evaluate this new integral.

Write $\frac{8u}{(u-1)(u+1)^2} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{(u+1)^2}$. Then clearing fractions gives

$$8u = A(u+1)^2 + B(u-1)(u+1) + C(u-1).$$

Selecting values for u now allows us to solve for A, B, and C:

$$u = 1: \quad \Rightarrow 8 = A(1+1)^2 \Rightarrow A = 2$$
$$u = -1: \quad \Rightarrow -8 = C(-1-1) \Rightarrow C = 4$$

Now select any other value of u and use the values of A and C to find B:

$$u = 0: \quad 0 = 2(0+1)^2 + B(0-1)(0+1) + 4(0-1) \Rightarrow 2 - B - 4 = 0 \Rightarrow B = -2$$

So

$$\frac{8u}{(u-1)(u+1)^2} = \frac{2}{u-1} - \frac{2}{u+1} + \frac{4}{(u+1)^2}$$

and

$$\int \frac{8u}{(u-1)(u+1)^2}\,\mathrm{d}u = \int \frac{2}{u-1}\,\mathrm{d}u - \int \frac{2}{u+1}\,\mathrm{d}u + \int \frac{4}{(u+1)^2}\,\mathrm{d}u$$

$$= 2\ln|u-1| - 2\ln|u+1| - 4(u+1)^{-1} + C$$

3. Consider the following integral. $$\int \cos^3(x)\sin^4(x)\ \mathrm{d}x$$

(a) Evaluate $\int \cos^3(x)\sin^4(x)\ \mathrm{d}x$

$$\int \cos^3(x)\sin^4(x)\ \mathrm{d}x = \int \cos^2(x)\sin^4(x)\cos(x)\ \mathrm{d}x$$

$$= \int [1-\sin^2(x)]\sin^4(x)\cos(x)\ \mathrm{d}x$$

Let $u = \sin(x) \Rightarrow \mathrm{d}u = \cos(x)\ \mathrm{d}x$

$$\int [1-\sin^2(x)]\sin^4(x)\cos(x)\ \mathrm{d}x = \int [1-u^2]u^4\ \mathrm{d}u = \int u^4 - u^6\ \mathrm{d}u$$

$$= \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{[\sin(x)]^5}{5} - \frac{[\sin(x)]^7}{7} + C$$

(b) Suppose a pdf is given by

$$f(x,y) = \begin{cases} c\cdot\cos^3(x)\sin^4(x) & \text{if } \ 0 \le x \le \dfrac{\pi}{2} \\ 0 & \text{otherwise.} \end{cases}$$

where c is some constant number.

Use properties of a pdf and your answer from part (a) to find the value for c.

For any pdf $f(x)$ we must have

$$\int_{-\infty}^{\infty} f(x)\ \mathrm{d}x = 1$$

Since our functions is zero over the intervals $(-\infty, 0)$ and $\left(\frac{\pi}{2}, \infty\right)$ this means

$$1 = \int_0^{\pi/2} f(x)\ \mathrm{d}x = \int_0^{\pi/2} c\cdot\cos^3(x)\sin^4(x)\ \mathrm{d}x$$

According to the answer above,

$$\int_0^{\pi/2} \cos^3(x)\sin^4(x)\ \mathrm{d}x = \left(\frac{[\sin(x)]^5}{5} - \frac{[\sin(x)]^7}{7}\right)\Bigg|_0^{\pi/2} = \frac{1}{5} - \frac{1}{7} = \frac{2}{35}$$

Then

$$1 = \int_0^{\pi/2} c\cdot\cos^3(x)\sin^4(x)\ \mathrm{d}x = c\cdot\frac{2}{35}$$

implies that $c = \dfrac{35}{2} = 17.5$.

4. The following questions have to do with the integral $I = \int \sin(2x) \cdot \cos(3x)\ \mathrm{d}x$

(a) Demonstrate the results of one application of integration by parts applied to $\int \cos(3x) \cdot \cos(x)\ \mathrm{d}x$.

$$\begin{aligned} I = \int \cos(3x)\cos(x)\ \mathrm{d}x &= \int \cos(3x)(\sin(x))'\ \mathrm{d}x \\ &= \sin(x)\cos(3x) - \int \sin(x)(\cos(3x))'\ \mathrm{d}x \\ &= \sin(x)\cos(3x) - 3\int \sin(x)\sin(3x)\ \mathrm{d}x \end{aligned}$$

(b) Continue your work from part (a) and evaluate $I = \int \cos(3x) \cdot \cos(x)\ \mathrm{d}x$.

$$\begin{aligned} I &= \sin(x)\cos(3x) - 3\int \sin(x)\sin(3x)\ \mathrm{d}x \\ &= \sin(x)\cos(3x) - 3\int (-\cos(x))'\sin(3x)\ \mathrm{d}x \\ &= \sin(x)\cos(3x) - 3[-\cos(x)\sin(3x) - \int -\cos(x)(\sin(3x))'\ \mathrm{d}x] \\ &= \sin(x)\cos(3x) - 3[-\cos(x)\sin(3x) + 3\int \cos(x)\cos(3x)\ \mathrm{d}x] \\ &= \sin(x)\cos(3x) + 3\cos(x)\sin(3x) - 9\int \cos(x)\cos(3x)\ \mathrm{d}x] \\ &= \sin(x)\cos(3x) + 3\cos(x)\sin(3x) - 9I \end{aligned}$$

Then $I = \sin(x)\cos(3x) - 3\cos(x)\sin(3x) - 9I$ implies that $10I = \sin(x)\cos(3x) + 3\cos(x)\sin(3x)$ or that

$$I = \frac{1}{10}[\sin(x)\cos(3x) + 3\cos(x)\sin(3x)] + C.$$

5. Consider the function $f(x) = xe^{-x}$

(a) Find a relevant Taylor Polynomial that will allow you to evaluate the limit.

We should use $a = 0$ since the limit is as $x \to 0$. We will need at least $P_2(x)$ since there is a x^2 in the denominator. The appropriate derivatives and values are given in the chart below:

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-x}	0
1	$-xe^{-x} + e^{-x} = (1-x)e^{-x}$	1
2	$-e^{-x} - (1-x)e^{-x} = (x-2)e^{-x}$	-2

Assembling the pieces now gives

$$P_2(x) = f^{(0)}(0) + \frac{f^{(1)}(0)}{1!}(x-0) + \frac{f^{(2)}(0)}{2!}(x-0)^2 = x - \frac{2}{2}x^2 = x - x^2.$$

(b) Use your Taylor Polynomial answer to evaluate the following limit:

$$\lim_{x\to 0} \frac{x - xe^{-x}}{x^2}$$

From part (a) we see that

$$\lim_{x\to 0} \frac{x - xe^{-x}}{x^2} = \lim_{x\to 0} \frac{x - (x - x^2)}{x^2} = \lim_{x\to 0} \frac{x^2}{x^2} = \lim_{x\to 0} 1 = 1.$$