Math 106: Review for Exam II

INTEGRATION TIPS

- Substitution: usually let $u =$ a function that's "inside" another function, especially if du (possibly off by a multiplying constant) is also present in the integrand.

- Parts: $\int u\,dv = uv - \int v\,du$ or $\int uv'\,dx = uv - \int u'v\,dx$

  How to choose which part is u? Let u be the part that is higher up in the **LIATE** mnemonic below. (The mnemonics **ILATE** and **LIPET** will work equally well if you have learned one of those instead; in the latter **A** is replaced by **P**, which stands for "polynomial.")

  Logarithms (such as $\ln x$)

  Inverse trig (such as $\arctan x, \arcsin x$)

  Algebraic (such as $x, x^2, x^3 + 4$)

  Trig (such as $\sin x, \cos 2x$)

  Exponentials (such as e^x, e^{3x})

- Rational Functions (one polynomial divided by another): if the degree of the numerator is greater than or equal to the degree of the denominator, do long division then integrate the result.

  Partial Fractions: here's an illustrative example of the setup.

$$\frac{3x^2+11}{(x+1)(x-3)^2(x^2+5)} = \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2} + \frac{Dx+E}{x^2+5}$$

  Each linear term in the denominator on the left gets a constant above it on the right; the squared linear factor $(x-3)$ on the left appears twice on the right, once to the second power. Each irreducible quadratic term on the left gets a linear term ($Dx + E$ here) above it on the right.

- Trigonometric Substitutions: some suggested substitutions and useful formulae follow.

Radical Form	$\sqrt{a^2-x^2}$	$\sqrt{a^2+x^2}$	$\sqrt{x^2-a^2}$
Substitution	$x = a\sin t$	$x = a\tan t$	$x = a\sec t$

$$\sin^2 x + \cos^2 x = 1 \qquad \tan^2 x + 1 = \sec^2 x$$

$$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2} \qquad \cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$$

$$\sin(2x) = 2\sin x\cos x$$

- Powers of Trigonometric Functions: here are some strategies for dealing with these.

$\int \sin^m x \cos^n x\ dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sin x$ and substitute $u = \cos x$.	$\sin^2 x = 1 - \cos^2 x$
n odd	Break off one factor of $\cos x$ and substitute $u = \sin x$.	$\cos^2 x = 1 - \sin^2 x$
m even AND n even	Use $\sin^2 x + \cos^2 x = 1$ to reduce to only powers of $\sin x$ or only powers of $\cos x$, then use table of integrals #39–42 or identities shown to right of this box.	$\sin^2 x = \frac{1}{2} - \frac{\cos(2x)}{2}$ $\cos^2 x = \frac{1}{2} + \frac{\cos(2x)}{2}$

$\int \tan^m x \sec^n x \; dx$	Possible Strategy	Identity to Use
m odd	Break off one factor of $\sec x \tan x$ and substitute $u = \sec x$.	$\tan^2 x = \sec^2 x - 1$
n even	Break off one factor of $\sec^2 x$ and substitute $u = \tan x$.	$\sec^2 x = \tan^2 x + 1$
m even AND n odd	Use identity at right to reduce to powers of $\sec x$ alone. Then use table of integrals #51 or integration by parts.	$\tan^2 x = \sec^2 x - 1$

Useful Trigonometric Derivatives and Antiderivatives

$\frac{d}{dx} \tan x = \sec^2 x$ $\quad$ $\frac{d}{dx} \sec x = \sec x \tan x$ $\quad$ $\int \sec x \; dx = \ln|\sec x + \tan x| + C$

- Improper integrals: look for ∞ as one of the limits of integration; look for functions that have a vertical asymptote in the interval of integration. It may be useful to know the following limits.

  $\lim_{x \to \infty} e^x =$

  $\lim_{x \to \infty} e^{-x} =$ $\quad$ Note: this is the same as $\lim_{x \to -\infty} e^x$

  $\lim_{x \to \infty} 1/x =$ $\quad$ Note: the answer is the same for $\lim_{x \to \infty} 1/x^2$ and similar functions

  $\lim_{x \to 0^+} 1/x =$ $\quad$ Note: the answer is the same for $\lim_{x \to 0^+} 1/x^2$ and similar functions

  $\lim_{x \to \infty} \ln x =$

  $\lim_{x \to 0^+} \ln x =$

  $\lim_{x \to \infty} \arctan x =$

1. Evaluate the following.

   (a) $\int \sin^6 x \cos^3 x \, dx$

   (b) $\int \frac{dx}{\sqrt{100 + x^2}}$

(c) $\displaystyle\int_3^\infty \frac{1}{x(\ln x)^{100}}\,dx$

(d) $\displaystyle\int_0^\infty xe^{-2x}\,dx$

(e) $\displaystyle\int \frac{3x^2+2x-13}{(x-3)(x^2+1)}\,dx$

(f) $\displaystyle\int \frac{4x^3-27x^2+20x-17}{x-6}\,dx$

(g) $\displaystyle\int_{-1}^5 \frac{1}{(x-1)^6}\,dx$

2. Find the second-order Taylor polynomial for $f(x) = \sqrt{x}$ based at $x_0 = 100$. Then use your polynomial to estimate $\sqrt{105}$.

3. What is the largest possible error that could have occurred in your estimate of $\sqrt{105}$?

4. Use comparisons to show whether each of the following converges or diverges. If an integral converges, also give a good upper bound for its value.

   (a) $\displaystyle\int_1^\infty \frac{6 + \cos x}{x^{0.99}}\, dx$

   (b) $\displaystyle\int_1^\infty \frac{4x^3 - 2x^2}{2x^4 + x^5 + 1}\, dx$

5. The probability density function (pdf) of the length (in minutes) of phone calls on a wireless network is given by $f(x) = ke^{-0.2x}$ where x is the number of minutes. Note that the domain is $x \geq 0$ since we can't have a negative number of minutes.

   (a) What must be the value of k?

   (b) What fraction of calls last more than 3 minutes?