Math 105
Exam 1
10/5/12

Name SOLUTIONS

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit.

1. (20 pts) Find the derivative for each of the following functions:

   (a) $f(t) = 2t^5 - \sqrt[3]{t^2} - 17$

   $f(t) = 2t^5 - t^{2/3} - 17$

   $f'(t) = 10t^4 - \frac{2}{3}t^{-1/3} = \boxed{10t^4 - \frac{2}{3\sqrt[3]{t}}}$

   (b) $g(x) = ax^4 + bx^2 - cx + d$, where $a, b, c,$ and d are constants.

   $g'(x) = \boxed{4ax^3 - 2bx - c}$

   (c) $h(u) = \dfrac{1-u}{u^3}$

   $h(u) = \frac{1}{u^3} - \frac{u}{u^3} = u^{-3} - u^{-2}$

   $h'(u) = -3u^{-4} + 2u^{-2} = \boxed{\frac{-3}{u^4} + \frac{2}{u^2}}$

2. (15 pts) Find the solution to the initial value problem: $y' = -\dfrac{2}{x^2} + 3$ when $y(1) = 4$.

   $y' = -2x^{-2} + 3$

   $y = \frac{-2x^{-1}}{-1} + 3x + c \Rightarrow y = \frac{2}{x} + 3x + c$

   Use $y(1)=4$ to find c

   $4 = \frac{2}{1} + 3(1) + c \Rightarrow c = -1$

   $\boxed{y = \frac{2}{x} + 3x - 1}$

3. (15 pts) Use the definition of the derivative as a limit to show $\frac{d}{dx}(5-4x-3x^2) = -4-6x$.

let $f(x) = 5-4x-3x^2$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{[5-4(x+h)-3(x+h)^2]-(5-4x-3x^2)}{h}$$

$$= \lim_{h\to 0} \frac{5-4x-4h-3x^2-6xh-3h^2-5+4x+3x^2}{h}$$

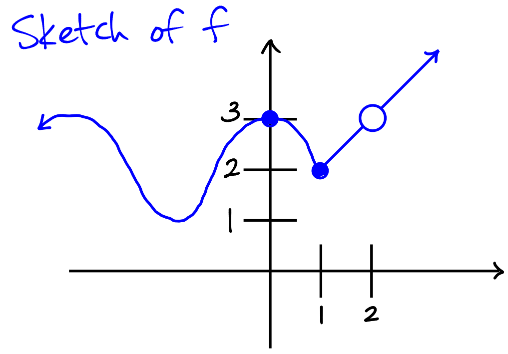
$$= \lim_{h\to 0} \frac{-4h-6xh-3h^2}{h}$$

$$= \lim_{h\to 0} \frac{h(-4-6x-3h)}{h}$$

$$= \lim_{h\to 0} (-4-6x-3h)$$

$$= -4-6x-0$$

$$f'(x) = -4-6x$$



4. (30 pts) Consider $f(x) = \begin{cases} 2+\cos x, & \text{if } x \le 0 \\ 3 - x^2, & \text{if } 0 < x \le 1 \\ \dfrac{x^2-x-2}{x-2}, & \text{if } x > 1 \end{cases}$

Determine the following values (if they exist)

(a) $\lim_{x\to 0^-} f(x)$

$\lim_{x\to 0^-} 2+\cos x = 2+\cos(0) = 3$

(b) $\lim_{x\to 0^+} f(x)$

$\lim_{x\to 0^+} 3-x^2 = 3-0 = 3$

(c) $f(0)$

$f(0) = 2+\cos(0) = 3$

(d) $\lim_{x\to 2} f(x)$

$\lim_{x\to 2} \frac{x^2-x-2}{x-2} = \lim_{x\to 2} \frac{(x+1)(x-2)}{x-2} = \lim_{x\to 2} x+1 = 3$

(e) $\lim_{x\to 1^-} f(x)$

$\lim_{x\to 1^-} 3-x^2 = 3-(1)^2 = 2$

(f) $\lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^+} \frac{x^2-x-2}{x-2} = \lim_{x\to 1^+} x+1 = 2$

(g) $\lim_{x\to 1} f(x)$

$\lim_{x\to 1} f(x) = 2$ since left & right hand limits are equal to 2

(h) $f(1)$

$f(1) = 3-(1)^2 = 2$

(i) Is f continuous at $x = 0$? Explain your answer using limits.

Yes, since $\lim_{x\to 0} f(x) = 3 = f(0)$

(j) Is f continuous at $x = 2$? Explain your answer.

No, there is a hole in the graph of f at $x=2$ since $\lim_{x\to 2} f(x) = 3$ but $f(2)$ does not exist

(k) Does $f'(0)$ exist? Explain your answer. (Hint: consider the graph of f.)

Yes, $f'(0) = 0$ since the slope of f goes to 0 as $x\to 0^-$ and $x\to 0^+$

(l) Does $f'(1)$ exist? Explain your answer. (Hint: consider the graph of f.)

No, the graph of f has a sharp corner at $x=1$

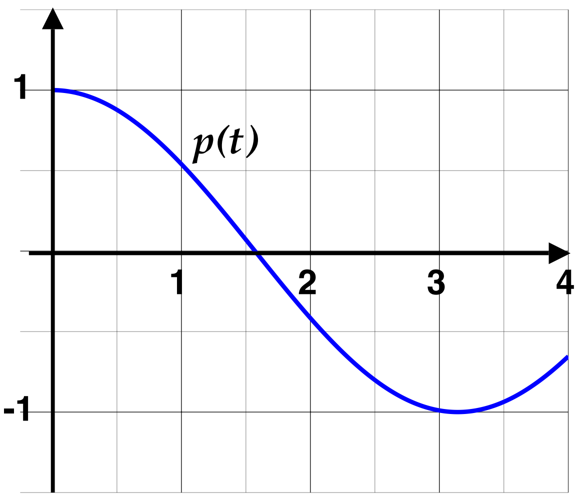
(In particular, as $x\to 1^-$, slope of f $\to -2$; but as $x\to 1^+$, slope of f $\to 1$)

(m) What is the range of f?

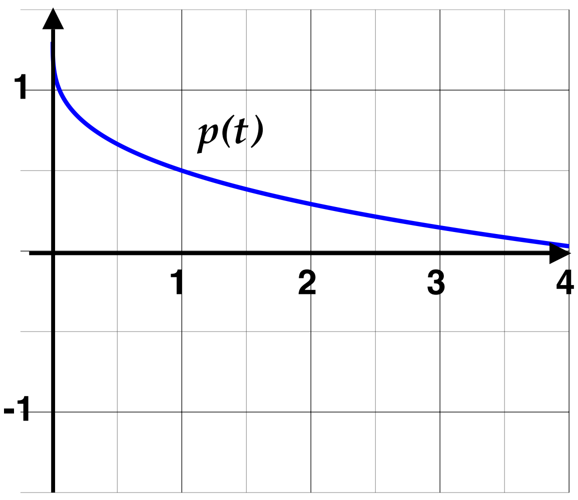
$[1, \infty)$

5. (20 pts) Each of the graphs below shows the position $p(t)$ of an object moving along the x-axis as a function of time t for $0 \leq t \leq 4$. When $p(t) < 0$, the object is to the left of the origin and when $p(t) > 0$, the object is to the right of the origin.

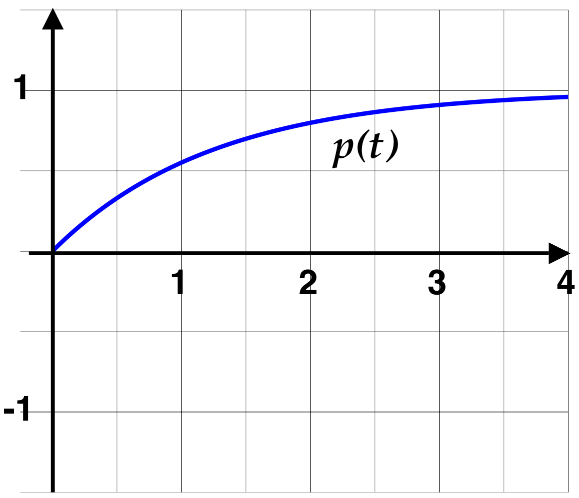
Graph A



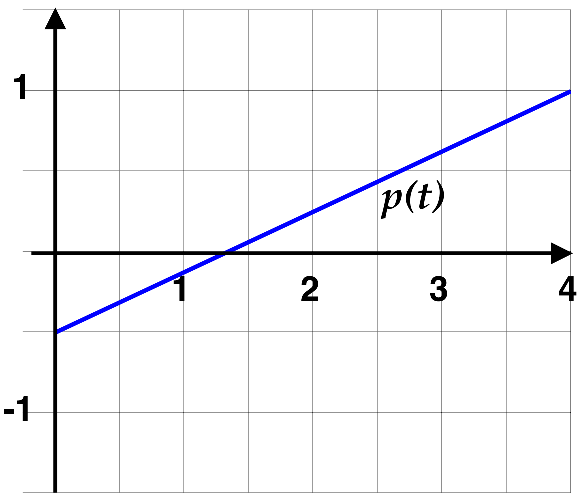
Graph B



Graph C



Graph D



On the interval $[0, 4]$, which graph(s) depicts an object that

(a) has constant velocity?

D, since the slope of p is constant.

(b) has positive acceleration over the entire interval?

B, since $a > 0 \Rightarrow p'' > 0 \Rightarrow p$ is concave up on the entire interval.

(c) has average velocity equal to 0.375?

D, since $\frac{p(4) - p(0)}{4 - 0} = \frac{1 - (-\frac{1}{2})}{4} = 0.375$

(d) has greatest initial speed?

B, since slope is steepest at left endpt of the interval.

(e) has zero acceleration?

D, since $a = 0 \Rightarrow v$ is constant $\Rightarrow p$ is a linear function.

(f) has decreasing velocity over the entire interval?

C, since v is decreasing when $v' = a < 0 \Rightarrow p'' = a < 0 \Rightarrow p$ is concave down on entire interval.

(g) changes direction?

A, since $p' = v$ changes sign (from negative to positive near $x \approx 3.2$).