MATH106B,C CALCULUS II - PROF. P. WONG

EXAM I - SEPTEMBER 27, 2013

NAME:

Instruction: Read each question carefully. Explain **ALL** your work and give reasons to support your answers.

Advice: DON'T spend too much time on a single problem.

Problems	**Maximum Score**	**Your Score**
1.	20	
2.	20	
3.	20	
4.	22	
5.	18	
Total	100	

1.(10 pts.)(a) Evaluate the indefinite integral (be sure to show all your work)

$$\int (\sin x)e^{\cos x}\ dx.$$

Let $u = \cos x$**. It follows that** $du = -\sin x\ dx$ **or** $\sin x\ dx = -du$**. Now we have**

$$\begin{aligned}\int (\sin x)e^{\cos x}\ dx &= \int e^u(-du)\\ &= -\int e^u\ du = -e^u + C\\ &= -e^{\cos x} + C.\end{aligned}$$

(10 pts.) (b) Find the **exact value** of the definite integral (be sure to show all your work)

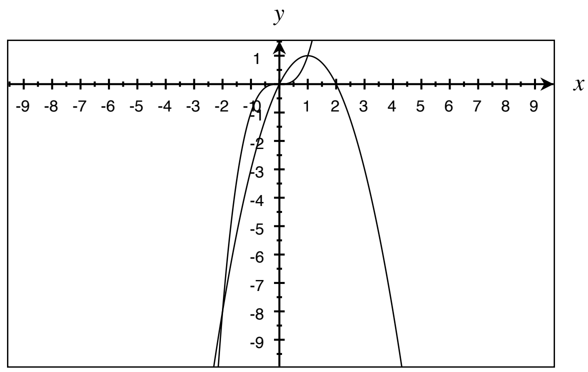
$$\int_0^2 \frac{1+e^x}{x+e^x}\ dx.$$

Let $w = x + e^x$ **so that** $dw = (1 + e^x)\ dx$**. When** $x = 0, w = 0 + e^0 = 1$**. When** $x = 2, w = 2 + e^2$**. It follows that**

$$\begin{aligned}\int_0^2 \frac{1+e^x}{x+e^x}\ dx &= \int_1^{2+e^2} \frac{dw}{w}\\ &= \ln|w|\Big|_1^{2+e^2}\\ &= \ln(2+e^2) - \ln 1 = \ln(2+e^2).\end{aligned}$$

2. Consider the region A in the first quadrant bounded by the curve $y = x^3$ and the curve $y = x(2-x)$.

(15 pts.) Find the **exact area** of the region A.



In the first quadrant, the cubic $y = x^3$ and the parabola $x(2-x)$ intersect at $(0,0)$ and $(1,1)$. By using vertical slices, the area of region A is given by

$$\textbf{area of } A = \int_0^1 x(2-x) - x^3 \, dx$$
$$= \int_0^1 2x - x^2 - x^3 \, dx = x^2 - \frac{x^3}{3} - \frac{x^4}{4}\Big|_0^1$$
$$= 1 - \frac{1}{3} - \frac{1}{4} = \frac{5}{12}.$$

Alternatively, if we use horizontal slices, we have

$$\textbf{area of } A = \int_0^1 y^{1/3} - (1 - \sqrt{1-y}) \, dy$$
$$= \frac{y^{4/3}}{4/3} - y - \frac{(1-y)^{3/2}}{3/2}\Big|_0^1$$
$$= (\frac{3}{4} - 1) - (-\frac{2}{3}) = \frac{5}{12}.$$

(5 pts.) The same two curves $y = x^3$ and $y = x(2-x)$ also bound a region B in the third quadrant. Write a definite integral (do not evaluate) representing the area of region B. [By area, we mean the usual *physical* area not *signed* area.]

In the third quadrant, the two curves meet at $(0,0)$ and $(-2,-8)$. The *physical* area of region B is given by

$$-\int_{-2}^0 x(2-x) - x^3 \, dx = \int_{-2}^0 x^3 - x(2-x) \, dx.$$

3. (12 pts.) Consider a function h on the interval $[0, 2]$.

x	0	0.5	1	1.5	2
$h(x)$	-1	1	3	2	-1

Find L_4, T_4 using the left-hand sum and the trapezoid rule respectively for estimating the definite integral $\int_0^2 h(x)\ dx$.

Here $\Delta x = 0.5$ so that

$$\begin{aligned} L_4 &= [h(0) + h(0.5) + h(1) + h(1.5)] \cdot \Delta x \\ &= [(-1) + (1) + (3) + (2)] \cdot (0.5) = 2.5. \end{aligned}$$

Similarly, we have

$$\begin{aligned} R_4 &= [h(0.5) + h(1) + h(1.5) + h(2)] \cdot \Delta x \\ &= [(1) + (3) + (2) + (-1)] \cdot (0.5) = 2.5. \end{aligned}$$

It follows that

$$T_4 = \frac{L_4 + R_4}{2} = \frac{2.5 + 2.5}{2} = 2.5.$$

(8 pts.)(b) Recall that the error committed by using the right hand sum approximation R_n is less than or equal to $\frac{K_1 \cdot (b-a)^2}{2n}$ where $|f'(x)| \leq K_1$ for some constant K_1 over the interval $[a, b]$. Use this result to give an upper bound for the error committed by R_{10} for

$$I = \int_1^3 (\sin x)(\ln x)\ dx.$$

In order to find K_1, first we find $f'(x)$. Here, $f(x) = (\sin x)(\ln x)$, it follows from the product rule that

$$f'(x) = (\cos x)(\ln x) + (\sin x) \cdot \frac{1}{x}.$$

Over the interval $[1, 3]$, one can use a graphing calculator to see that f' is decreasing and that the maximum of the absolute value $|f'(x)|$ actually occurs at $x = 3$ so that $|f'(x)| \leq 1.041$. Choose $K_1 = 1.041$.

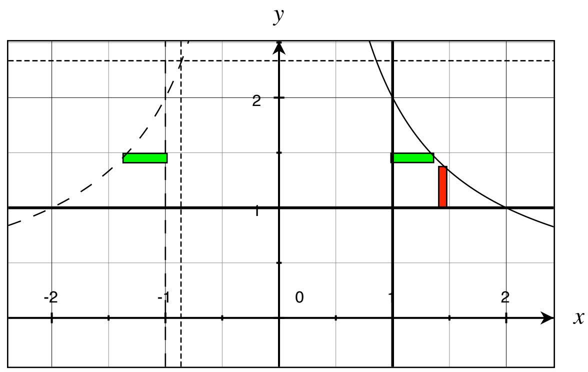
Now, the error committed by R_{10} is

$$\frac{(1.041)(3-1)^2}{20} = 0.2082.$$

If you do not use a graphing calculator, we may estimate K_1 as follows. Since $|cosx| \leq 1$ and $|\sin x| \leq 1$, we conclude that $|f'(x)| \leq |lnx| + \frac{1}{x}$. Over the interval $[1, 3]$, $\ln x \leq \ln 3$ and $\frac{1}{x} \leq 1$. Thus, $|f'(x)| \leq 1 + \ln 3$ or we can choose $K_1 = 1 + \ln 3$. We then proceed the error esitmation as before.

4. Let R be the region bounded by the curve $y = \frac{2}{x}$, the line $y = 1$, the line $y = 2$, and the line $x = 1$.

(12 pts.) (a) Set up (do not evaluate) a definite integral representing the volume of the solid obtained from rotating the region R around the line $x = 0$, i.e., the y-axis. [Hint: sketch a picture of the region R first.]



If we use horizontal slices, then a typical slice (green) looks like a "washer" with thickness Δy. Thus, the volume of the solid is given by

$$\int_1^2 \pi(\frac{2}{y})^2 - \pi(1)^2 \, dy.$$

If you use vertical slices, then a typical slice (red) looks like a "cylindrical shell" with thickness Δx. Thus, the volume of the solid is given by

$$\int_1^2 2\pi(x) \cdot (\frac{2}{x} - 1) \, dx.$$

(10 pts.) (b) Find the **exact volume** of the solid described in part (a).

From part(a), the volume is

$$\begin{aligned}\int_1^2 \pi\left(\frac{2}{y}\right)^2 - \pi(1)^2 \, dy &= \pi \int_1^2 4y^{-2} - 1 \, dy \\ &= \pi\left(\frac{4y^{-1}}{-1} - y\right)\Big|_1^2 \\ &= \pi[(-2-2) - (-4-1)] = \pi[-4+5] = \pi.\end{aligned}$$

5. Consider the initial value problem

$$\frac{dy}{dx} = (1 + y^2)e^x$$

with $y(0) = 0$.

(10 pts.)(a) Use the technique of separation of variables to solve the Initial Value Problem.

Rewriting the differential equation by separating the variables, we have

$$\int \frac{dy}{1 + y^2} = \int e^x \; dx.$$

It follows that

$$\arctan y = e^x + C. \tag{1}$$

Since $y(0) = 0$ **and** $\arctan 0 = 0$**,** (1) **becomes** $0 = e^0 + C$ **so** $C = -1$**. Now,** (1) **becomes** $\arctan y = e^x - 1$ **or**

$$y = \tan(e^x - 1).$$

(8 pts.)(b) Set up (do not evaluate) a definite integral for the arc length of the portion of the graph of $f(x) = x \ln x$ between $x = 1$ and $x = e$.

Note that $f'(x) = \ln x + x \cdot \frac{1}{x} = 1 + \ln x$**. It follows that the arc length is given by the following definite integral**

$$\int_1^e \sqrt{1 + (1 + \ln x)^2} \; dx.$$