Math 106 Fall 2013

Test 1 (50 points)

Name: Solutions.

Show all your work to receive full credit for a problem. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Please keep your written answers brief; be clear and to the point. Points will be taken off for rambling and for incorrect or irrelevant statements.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals.

When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

Include units in your answers wherever possible.

There are six questions. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$$|I-L_n| \le \frac{K_1(b-a)^2}{2n} \qquad |I-R_n| \le \frac{K_1(b-a)^2}{2n}$$

$$|I-T_n| \le \frac{K_2(b-a)^3}{12n^2} \qquad |I-M_n| \le \frac{K_2(b-a)^3}{24n^2}$$

Below are product rule, quotient rule and chain rule for derivatives.

$$(uv)' = u'v+uv' \qquad \left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2} \qquad (f\circ g)'(x) = f'(g(x))\,g'(x)$$

1. (8 points) Evaluate the following integral exactly. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\int \frac{5x^3}{x^2+9}\,dx\,.$$

$u = x^2+9$.

$\frac{du}{dx} = 2x \qquad dx = \frac{du}{2x}$.

$$\int \frac{5x^3}{x^2+9}\,dx = 5\int x^3 \cdot \frac{1}{x^2+9}\cdot dx$$

$$= 5\int x^2 \cdot x\,\frac{1}{x^2+9}\cdot dx$$

$$= 5\int x^2 \cdot \cancel{x}\,\frac{1}{u}\cdot\frac{du}{2\cancel{x}}$$

$u = x^2+9$

So $x^2 = u-9$

$$= 5\int (u-9)\cdot\frac{1}{u}\cdot\frac{du}{2}$$

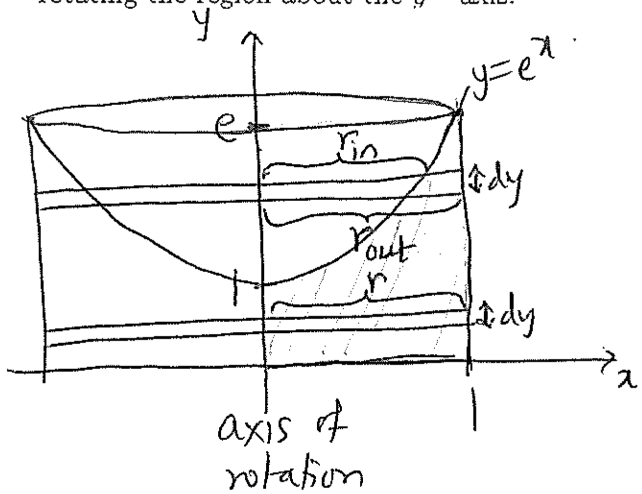
$$= \frac{5}{2}\int\left(1-\frac{9}{u}\right)du$$

$$= \frac{5}{2}\left(u - 9\ln|u|\right) + C.$$

$$= \frac{5}{2}\left((x^2+9) - 9\ln|x^2+9|\right) + C\,.$$

2. (9 points)

(a) Sketch the region bounded by the curve $y = e^x$, and the lines $x = 0$, $x = 1$ and $y = 0$. Write (but do not evaluate) an integral to find the volume of the solid obtained by rotating the region about the $y-$ axis.



For $1 < y \le e$,

slice is a washer.

$r_{in} = x = \ln y$ ($y = e^x$ so $x = \ln y$).

$r_{out} = 1$.

$\text{Area} = \pi r_{out}^2 - \pi r_{in}^2 = \pi(1)^2 - \pi(\ln y)^2$.

For $0 < y \le 1$,

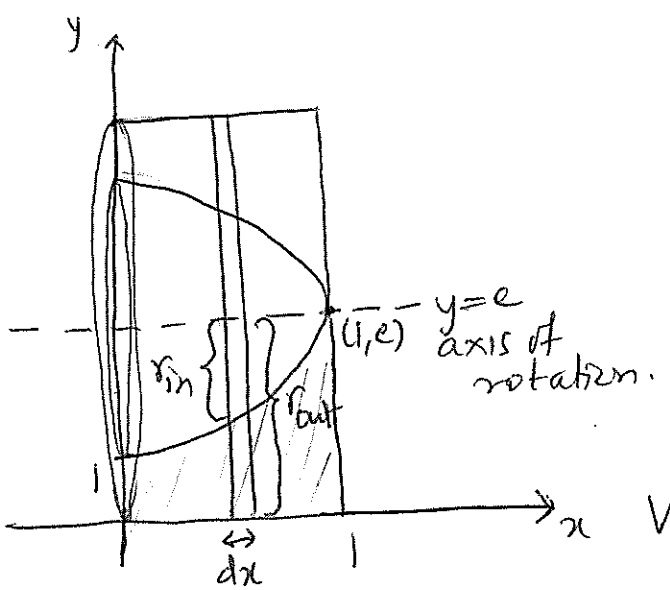
slice is a disk.

$r = 1$.

$\text{Area} = \pi r^2 = \pi(1)^2 = \pi$.

$$\text{Volume} = \int_0^1 \pi\, dy + \int_1^e (\pi - \pi(\ln y)^2)\, dy.$$

(b) Sketch the region bounded by the curve $y = e^x$, and the lines $x = 0$, $x = 1$ and $y = 0$. Write (but do not evaluate) an integral to find the volume of the solid obtained by rotating the region about the line $y = e$.



Slice is a washer.

$r_{in} = e - y = e - e^x$

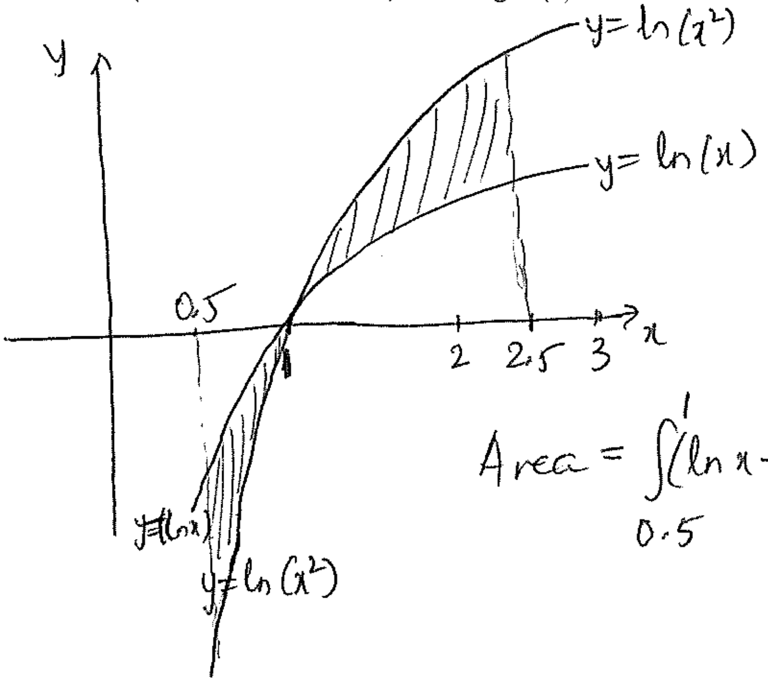
$r_{out} = e$.

$\text{Area} = \pi r_{out}^2 - \pi r_{in}^2$

$= \pi e^2 - \pi(e - e^x)^2$.

$$\text{Volume} = \int_0^1 (\pi e^2 - \pi (e - e^x)^2)\, dx.$$

3. (5 points) Sketch the region bounded by the curves $y = \ln x$, $y = \ln(x^2)$, $x = 0.5$ and $x = 2.5$. Write (but do not evaluate) an integral(s) to find the area of the region you have sketched.



$\ln x = \ln(x^2)$, $x > 0$. ln x is not defined for $x \le 0$.

$e^{\ln x} = e^{\ln(x^2)}$

$x = x^2$

$x - x^2 = 0$.

$x(x-1) = 0$. $x = 1$.

$$\text{Area} = \int_{0.5}^{1} (\ln x - \ln(x^2))\,dx + \int_{1}^{2.5} (\ln(x^2) - \ln x)\,dx$$

4. (7 points) Find the exact length of the curve $y = \sqrt{9 - x^2}$ from $x = 0$ to $x = 3/\sqrt{2}$.

$$f(x) = \sqrt{9-x^2} = (9-x^2)^{1/2}.$$

$$f'(x) = \frac{1}{\cancel{2}}(9-x^2)^{-1/2} \cdot (-\cancel{2}x) = \frac{-x}{\sqrt{9-x^2}}.$$

$$\text{Length} = \int_0^{3/\sqrt{2}} \sqrt{1+(f'(x))^2}\,dx = \int_0^{3/\sqrt{2}} \sqrt{1+\frac{x^2}{9-x^2}}\,dx$$

$$= \int_0^{3/\sqrt{2}} \sqrt{\frac{9-\cancel{x^2}+\cancel{x^2}}{9-x^2}}\,dx = \int_0^{3/\sqrt{2}} \frac{3}{\sqrt{9-x^2}}\,dx$$

$$= 3\int_0^{3/\sqrt{2}} \frac{1}{\sqrt{9-x^2}}\,dx = 3\arcsin\left(\frac{x}{3}\right)\Bigg]_0^{3/\sqrt{2}}$$ Formula 15 with $a=3$.

$$= 3\arcsin\left(\frac{3}{\sqrt{2}\cdot 3}\right) - 3\arcsin(0).$$

$$= 3\arcsin\left(\frac{1}{\sqrt{2}}\right) = 3\cdot\frac{\pi}{4}.$$

5. (9 points) The net worth of a company changes at a rate given by $W'(t) = 0.03(W - 4000)$, where $W(t)$ million dollars is the net worth at t years.

(a) Suppose the company starts with a net worth of 3000 million dollars, i.e., $W = 3000$ when $t = 0$. Find an equation for the net worth of the company at time t.

$$W' = 0.03(W-4000)$$

$$\frac{dW}{dt} = 0.03(W-4000)$$

$$\int \frac{dW}{W-4000} = \int 0.03\,dt$$

For $\int \frac{dW}{W-4000}$, let $u = W-4000$

$\frac{du}{dW} = 1.\quad dW = du.$

$$\int \frac{dW}{W-4000} = \int \frac{du}{u} = \ln|u| = \ln|W-4000|$$

$$\ln|W-4000| = 0.03t + C.$$

$$|W-4000| = e^{0.03t+C}$$

$$W-4000 = \pm e^{C} \cdot e^{0.03t}$$

i.e. $W-4000 = Ae^{0.03t}$, $A = \pm e^{C}$.

$$W = 4000 + Ae^{0.03t}$$

When $t=0$, $W=3000$

$$3000 = 4000 + A$$

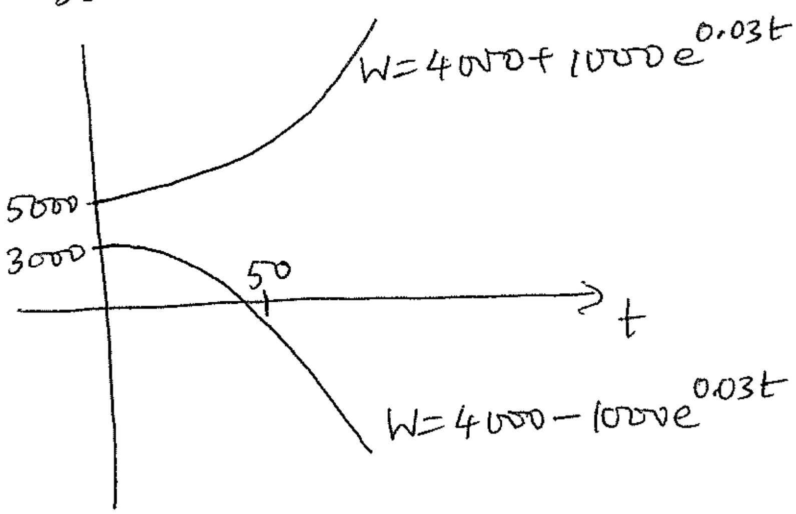
$$A = -1000$$

$$W = 4000 - 1000e^{0.03t}$$

(b) How does the equation you got in part (a) change if the company starts with a net worth of 5000 million dollars? Sketch this new equation for W and the equation from part (a) on the same set of axes. Comment in a sentence or two on the net worth of the company in the long run in the two cases, as seen in your graphs.

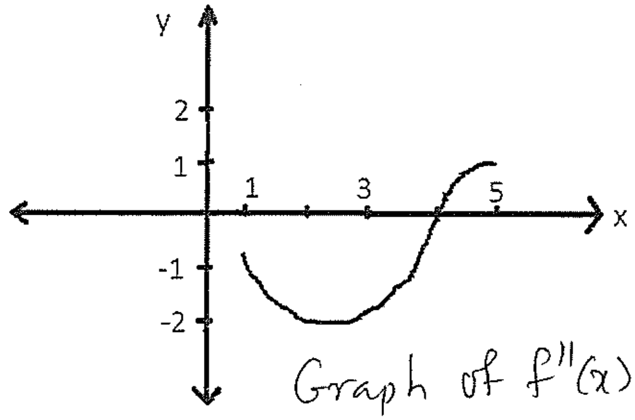
If $W = 5000$, when $t=0$, we get $5000 = 4000 + A$.

So $A = 1000$ and $W = 4000 + 1000e^{0.03t}$.

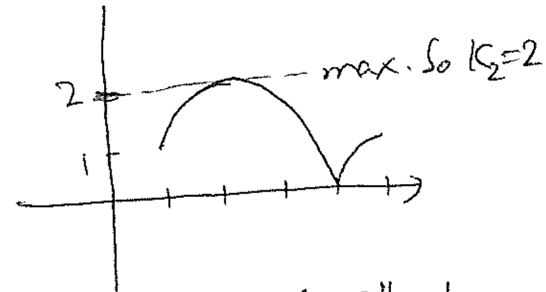


If the company starts with 5000 million dollars, then its net worth keeps on increasing in the long run but if it starts with 3000 million dollars, then its net worth keeps on decreasing in the long run and becomes negative eventually.

6. (12 points) The graph of the second derivative of a function f (i.e. the graph of $f''(x)$) on the interval $[1, 5]$ is given below. Suppose $f'(x)$ is positive on the interval $[1, 5]$. Use this information to answer the questions that follow. used in part (c).



Graph of $f''(x)$.



Graph of $|f''(x)|$.

(a) Let $I = \int_{1.5}^{4.5} f(x)\,dx$. What is the least value of n which guarantees that M_n approximates I within ± 0.001? Justify your answer.

We want $|I - M_n| \le 0.001$.

We know $|I - M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$

So if $\dfrac{K_2(b-a)^3}{24n^2} \le 0.001$, then we have $|I - M_n| \le 0.001$

From graph of $|f''(x)|$, $K_2 = 2$.

$a = 1.5$, $b = 4.5$

$\dfrac{2(3)^3}{24n^2} \le 0.001$.

i.e. $\dfrac{54}{24(0.001)} \le n^2$. So $n^2 \ge 2250$. i.e. $n \ge 47.434$

So $\boxed{n = 48}$.

(b) Does T_{35} overestimate or underestimate $\int_1^4 f(x)\,dx$? Justify your answer.

On $[1, 4]$, $f''(x) \le 0$ (as seen in graph).
So f is concave down.
So T_{35} underestimates $\int_1^4 f(x)\,dx$.

(c) Suppose you want to find the length of the curve $f(x)$ from $x = 1$ to $x = 4$. Does L_{35} overestimate or underestimate the definite integral you need to compute to find this length? Justify your answer.

Length $= \int_1^4 \sqrt{1+(f'(x))^2}\,dx$. Let $g(x) = \sqrt{1+(f'(x))^2} = (1+(f'(x))^2)^{1/2}$.

We need to find if $g(x)$ is increasing or decreasing on $[1, 4]$. $g'(x) = \dfrac{1}{\cancel{2}}(1+(f'(x))^2)^{-1/2} \cdot \cancel{2} f'(x) \cdot f''(x)$ (chain rule)

i.e. $g'(x) = \dfrac{\underbrace{f'(x)}_{\text{positive}}\ \underbrace{f''(x)}_{\text{negative}}}{\underbrace{\sqrt{1+(f'(x))^2}}_{\text{positive}}}$ So $g'(x) \le 0$ on $[1, 4]$.

Hence $g(x)$ is decreasing on $[1, 4]$. So L_{35} overestimates the length integral.