1. Answer the following questions about the definite integral $I = \int_{0.5}^{0.75} \left(x - \frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x$.

(a) Find the exact value of $I = \int_{0.5}^{0.75} \left(x - \frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x$.

You can write your final answer using the evaluation bar $(F(x))\Big|_{0.5}^{0.75}$.

$$\int \left(x - \frac{1}{\sqrt{x}}\right)^2 \mathrm{d}x = \int x^2 - 2\sqrt{x} + \frac{1}{x}\ \mathrm{d}x = \int x^2 - 2x^{1/2} + \frac{1}{x}\ \mathrm{d}x$$

$$= \frac{x^3}{3} - \frac{4}{3}x^{3/2} + \ln|x| + C$$

Now using FTOC, we have

$$\int_1^4 \left(\sqrt{x} + \frac{1}{x}\right)^2 \mathrm{d}x = \left(\frac{x^3}{3} - \frac{4}{3}x^{3/2} + \ln|x|\right)\Bigg|_{0.5}^{0.75}$$

(b) The function $f(x) = \left(x - \frac{1}{\sqrt{x}}\right)^2$ is decreasing and concave up on the interval $[0.5, 0.75]$. Which of the following pairs of approximations will give an overestimate of I? Circle one correct answer.

R_n and M_n $\qquad$ $L_n$and M_n $\qquad$ R_n and T_n $\qquad$ L_n and T_n

2. Answer the following questions related to $\int x^3\sqrt{x^2-1}\,\mathrm{d}x$

(a) Evaluate the indefinite integral $$\int x^3\sqrt{x^2-1}\,\mathrm{d}x$$

Let $u = x^2 - 1 \Leftrightarrow u + 1 = x^2 \Rightarrow \dfrac{\mathrm{du}}{\mathrm{dx}} = 2x \Rightarrow \dfrac{1}{2}\,\mathrm{d}u = x\,\mathrm{d}x$

$$\int x^3\sqrt{x^2-1}\,\mathrm{d}x = \int x^2\sqrt{x^2-1}\,x\,\mathrm{d}x = \frac{1}{2}\int (u+1)\sqrt{u}\,\mathrm{d}u = \frac{1}{2}\int u^{3/2}+u^{1/2}\,\mathrm{d}u$$

$$= \frac{1}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C = \frac{1}{5}(x^2-1)^{5/2} + \frac{1}{3}(x^2-1)^{3/2} + C$$

(b) For $I = \int_2^{11} x^3\sqrt{x^2-1}\,\mathrm{d}x$ find the following components needed to calculate R_{10}.

$\Delta x = \dfrac{b-a}{n} = \dfrac{11-2}{10} = \dfrac{9}{10} = 0.9$

$x_k = a + \Delta x \cdot k = 2 + 0.9k$

$f(x_k) = (2+0.9k)^3\sqrt{(2+0.9k)^2-1}$

3. The following questions are about $f(x) = x^{1/2} - \dfrac{x^{3/2}}{3}$

(a) Find a value K_1 that is appropriate for the R_n error bound formula

$$|I - R_n| \leq \frac{K_1(b-a)^2}{2n}$$

for $I = \displaystyle\int_4^9 x^{1/2} - \frac{x^{3/2}}{3}\,\mathrm{d}x$.

Note that
$f'(x) = \dfrac{1}{2}x^{-1/2} - \dfrac{1}{2}x^{1/2} = \dfrac{1}{2x^{1/2}} - \dfrac{1}{2}x^{1/2} = \dfrac{1-x}{2x^{1/2}}$

By graphing we can check that $|f'(x)|$ achieves a maximum on the interval $[4, 9]$ at $x = 9$. Then a K_1 value that works is

$$|f'(9)| = \left|\frac{1-9}{2 \cdot 9^{1/2}}\right| = 1.\overline{3} = K_1$$

(b) Set up and evaluate the integral for the arc length of $f(x) = x^{1/2} - \dfrac{x^{3/2}}{3}$ over the interval $4 \leq x \leq 9$.

You should write your final answer using the evaluation bar notation $\left.(F(x))\right|_4^9$.

This is an algebra problem:

$$f(x) = x^{1/2} - \frac{x^{3/2}}{3}$$

$$f'(x) = \frac{1}{2x^{1/2}} - \frac{1}{2}x^{1/2} = \frac{1-x}{2x^{1/2}}$$

$$\left[f'(x)\right]^2 = \frac{1-2x+x^2}{4x}$$

$$\left[f'(x)\right]^2 + 1 = \frac{1-2x+x^2}{4x} + \frac{4x}{4x} = \frac{1+2x+x^2}{4x}$$

$$\sqrt{\left[f'(x)\right]^2 + 1} = \sqrt{\frac{1+2x+x^2}{4x}} = \sqrt{\frac{(1+x)^2}{(2x^{1/2})^2}} = \frac{1+x}{2x^{1/2}}$$

So $\displaystyle L = \int_4^9 \frac{1+x}{2x^{1/2}}\,\mathrm{d}x = \int_4^9 \frac{1}{2x^{1/2}} + \frac{1}{2}x^{1/2}\,\mathrm{d}x = \int_4^9 \frac{1}{2}x^{-1/2} + \frac{1}{2}x^{1/2}\,\mathrm{d}x = \left.\left(x^{1/2} + \frac{1}{3}x^{3/2}\right)\right|_4^9$

4. The following questions have to do with the curves given by $y = x^4$ and $y = 2x^3$.

   (a) Use integrals to write an expression that computes the area for the region between the two curves over the interval $[1, 4]$. **DO NOT EVALUATE ANY INTEGRAL!**

   Sketch the region. You should notice that $y = 2x^3$ is the "Top" function on the interval $[1, 2]$ with $y = x^4$ the "Bottom" function. On the interval $[2, 4]$ these roles are reversed. Therefore, the area between the two curves is given by the integral

$$A = \int_1^2 2x^3 - x^4 \, dx + \int_2^4 x^4 - 2x^3 \, dx$$

   (b) Consider the two curves $y = x^4$ and $y = 2x^3$ and the region between where they cross. Write an integral for the volume of the solid made by rotating/revolving this region about the line $y = -1$. For partial credit, be sure to identify the appropriate interval, your r_o and r_i, and the integrand (inside of the integral). **DO NOT EVALUATE ANY INTEGRAL!**

   From the sketch above, you should see that $y = x^4$ and $y = 2x^3$ cross when

$$x^4 = 2x^3 \Rightarrow x^4 - 2x^3 = 0 \Rightarrow x^3(x - 2) = 0$$

   or when $x = 0$ and $x = 2$.

   The solid is obtained by rotation about $y = -1$, a line that is parallel to the x-axis. Therefore, our volume will be of the form

$$V = \int \qquad dx.$$

   Since we know the bounds for x of the crossings, we know that

$$V = \int_0^2 \qquad dx.$$

   To create the appropriate area function for each cross section at x, notice that a length of 1 is added to the height of our graphs to create the radii. Using the "Top" function of $y = 2x^3$ we have

$$r_0 = 1 + 2x^3$$

   and using the "Bottom" function of $y = x^4$ we have

$$r_i = 1 + x^4.$$

   Then $A(x) = \pi[r_o^2 - r_i^2] = \pi\left[(1 + 2x^3)^2 - (1 + x^4)^2\right]$ and

$$V = \int_0^2 \pi\left[(1 + 2x^3)^2 - (1 + x^4)^2\right] dx.$$

5. The following questions have to do with the effects of gravity in the Nintendo game *Super Mario Galaxy.* In that game the protagonist Mario explores the galaxy by jumping from planet to planet.

(a) In some parts of the game, the effect of gravity can by modeled by the Separable Differential Equation

$$y' = y^2(\sin x - e^{2x}).$$

Find an explicit solution to this Differential Equation.

Begin by separating the variables:

$$y' = y^2(\sin x - e^{2x}) \Rightarrow \frac{\mathrm{dy}}{\mathrm{dx}} = y^2(\sin x - e^{2x}) \Rightarrow \frac{1}{y^2}\ \mathrm{d}y = (\sin x - e^{2x})\ \mathrm{d}x$$

Now integrating gives the following:

$$\int y^{-2}\ \mathrm{d}y = \int \sin x - e^{2x}\ \mathrm{d}x$$

Most antiderivatives here are basic, but $\int e^{2x}\ \mathrm{d}x$ requires the substitution $u = 2x$ with $\frac{1}{2}\ \mathrm{d}x = \mathrm{d}u$ so that

$$\int e^{2x}\ \mathrm{d}x = \frac{1}{2}\int e^u\ \mathrm{d}u = \frac{1}{2}e^u = \frac{1}{2}e^{2x}.$$

After all integrations we have

$$-y^{-1} = -\cos x - e^{2x} + C.$$

Now solving for y gives

$$\begin{aligned} y^{-1} = \cos x + e^{2x} - C \Rightarrow \frac{1}{y} &= \cos x + e^{2x} - C \\ \Rightarrow y &= \frac{1}{\cos x + e^{2x} - C} \end{aligned}$$

(b) In other parts of the game *Super Mario Galaxy*, the force exerted when x units from a black hole is given by

$$F(x) = \frac{6}{x^2+1}.$$

Set up and evaluate the integral for the amount of work required for Mario to jump from a planet that is 4 units from the black hole to an above planet that is 8 units from the black hole.
You may write your final answer using the evaluation bar notation.

The work performed by Mario is given by:

$$W = \int_4^8 \frac{6}{x^2+1}\ \mathrm{d}x = 6\int_4^8 \frac{1}{x^2+1}\ \mathrm{d}x = (6\arctan x)\Bigg|_4^8$$