

1. Answer the following questions about the definite integral $I = \int_1^4 \left(1 + \frac{1}{x}\right)^2 dx$.

(a) Find the exact value of $I = \int_1^4 \left(1 + \frac{1}{x}\right)^2 dx$.

You can write your final answer using the evaluation bar $(F(x)) \Big|_1^4$.

Use algebra to solve the indefinite integral first, then use FTOC.

$$\begin{aligned} \int \left(1 + \frac{1}{x}\right)^2 dx &= \int 1 + \frac{2}{x} + \frac{1}{x^2} dx = \int 1 + 2 \cdot \frac{1}{x} + x^{-2} dx \\ &= x + 2 \ln|x| - x^{-1} + C \end{aligned}$$

Now using FTOC, we have

$$\int_1^4 \left(\sqrt{x} + \frac{1}{x}\right)^2 dx = (x + 2 \ln|x| - x^{-1}) \Big|_1^4$$

(b) The function $f(x) = \left(1 + \frac{1}{x}\right)^2$ is increasing and concave up on the interval $[1, 4]$. Which of the following pairs of approximations will give an overestimate of I ? Circle one correct answer.

R_n and M_n

L_n and M_n

R_n and T_n

L_n and T_n

2. Answer the following questions related to $\int \frac{x}{\sqrt[4]{x-1}} dx$

(a) Evaluate the indefinite integral $\int \frac{x}{\sqrt[4]{x-1}} dx$

Let $u = x - 1 \Leftrightarrow u + 1 = x \Rightarrow du = dx$.

Then

$$\begin{aligned} \int \frac{x}{\sqrt[4]{x-1}} dx &= \int \frac{u+1}{\sqrt[4]{u}} du = \int \frac{u+1}{u^{1/4}} du = \int \frac{u}{u^{1/4}} + \frac{1}{u^{1/4}} du \\ &= \int u^{3/4} + u^{-1/4} du \\ &= \frac{4}{7}u^{7/4} + \frac{4}{3}u^{3/4} + C \\ &= \frac{4}{7}(x-1)^{7/4} + \frac{4}{3}(x-1)^{3/4} + C \end{aligned}$$

(b) For $I = \int_3^5 \frac{x}{\sqrt[4]{x-1}} dx$ find the following components needed to calculate L_{10} .

$$\Delta x = \frac{b-a}{n} = \frac{5-3}{10} = \frac{2}{10} = 0.2$$

$$x_{k-1} = \frac{a + \Delta x \cdot (k-1)}{1} = \frac{3 + 0.2(k-1)}{1}$$

$$f(x_{k-1}) = \frac{(3 + 0.2(k-1))}{\sqrt[4]{3 + 0.2(k-1)} - 1}$$

3. The following questions are about $f(x) = \frac{x^3}{6} + \frac{1}{2x}$

(a) Find a value K_1 that is appropriate for the R_n error bound formula

$$|I - R_n| \leq \frac{K_1(b-a)^2}{2n}$$

$$\text{for } I = \int_2^3 \frac{x^3}{6} + \frac{1}{2x} \, dx.$$

Note that

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{x^4 - 1}{2x^2}$$

By graphing we can check that $|f'(x)|$ achieves a maximum on the interval $[2, 3]$ at $x = 3$. Then a K_1 value that works is

$$|f'(3)| = \left| \frac{3^4 - 1}{2 \cdot 3^2} \right| = 4.\bar{4} = K_1$$

(b) Set up and evaluate the integral for the arc length of $f(x) = \frac{x^3}{6} + \frac{1}{2x}$ over the interval $2 \leq x \leq 3$.

You should write your final answer using the evaluation bar notation $(F(x)) \Big|_2^3$.

This is an algebra problem:

$$f(x) = \frac{x^3}{6} + \frac{1}{2x}$$

$$f'(x) = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{x^4 - 1}{2x^2}$$

$$[f'(x)]^2 = \frac{x^8 - 2x^4 + 1}{4x^4}$$

$$[f'(x)]^2 + 1 = \frac{x^8 - 2x^4 + 1}{4x^4} + \frac{4x^4}{4x^4} = \frac{x^8 + 2x^4 + 1}{4x^4}$$

$$\sqrt{1 + [f'(x)]^2} = \sqrt{\frac{x^8 + 2x^4 + 1}{4x^4}} = \sqrt{\frac{(x^4 + 1)^2}{(2x^2)^2}} = \frac{x^4 + 1}{2x^2}$$

$$\text{So } L = \int_2^3 \frac{x^4 + 1}{2x^2} \, dx = \int_2^3 \frac{x^2}{2} + \frac{1}{2x^2} \, dx = \left(\frac{x^3}{6} - \frac{1}{2x} \right) \Big|_2^3$$

4. The following questions have to do with the curves given by $y = 3x^2$ and $y = x^3$.

- (a) Use integrals to write an expression that computes the area for the region between the two curves over the interval $[1, 5]$. **DO NOT EVALUATE ANY INTEGRAL!**

Sketch the region. You should notice that $y = 3x^2$ is the “Top” function on the interval $[1, 3]$ with $y = x^3$ the “Bottom” function. On the interval $[3, 5]$ these roles are reversed. Therefore, the area between the two curves is given by the integral

$$A = \int_1^3 3x^2 - x^3 \, dx + \int_3^5 x^3 - 3x^2 \, dx$$

- (b) Consider the two curves $y = 3x^2$ and $y = x^3$ and the region between where they cross. Write an integral for the volume of the solid made by rotating/revolving this region about the line $y = -2$. For partial credit, be sure to identify the appropriate interval, your r_o and r_i , and the integrand (inside of the integral). **DO NOT EVALUATE ANY INTEGRAL!**

From the sketch above, you should see that $y = 3x^2$ and $y = x^3$ cross when

$$3x^2 = x^3 \Rightarrow x^3 - 3x^2 = 0 \Rightarrow x^2(x - 3) = 0$$

or when $x = 0$ and $x = 3$.

The solid is obtained by rotation about $y = -2$, a line that is parallel to the x -axis. Therefore, our volume will be of the form

$$V = \int \quad dx.$$

Since we know the bounds for x of the crossings, we know that

$$V = \int_0^3 \quad dx.$$

To create the appropriate area function for each cross section at x , notice that a length of 2 is added to the height of our graphs to create the radii. Using the “Top” function of $y = 3x^2$ we have

$$r_o = 2 + 3x^2$$

and using the “Bottom” function of $y = x^3$ we have

$$r_i = 2 + x^3.$$

Then $A(x) = \pi[r_o^2 - r_i^2] = \pi [(2 + 3x^2)^2 - (2 + x^3)^2]$ and

$$V = \int_0^3 \pi [(2 + 3x^2)^2 - (2 + x^3)^2] \, dx.$$

5. The following questions have to do with the effects of gravity in the Nintendo game *Super Mario Galaxy*. In that game the protagonist Mario explores the galaxy by jumping from planet to planet.

- (a) In some parts of the game, the effect of gravity can be modeled by the Separable Differential Equation

$$y' = y^6(e^x + \cos 2x).$$

Find a general solution to this Differential Equation.

Begin by separating the variables:

$$y' = y^6(e^x + \cos 2x) \Rightarrow \frac{dy}{y^6} = (e^x + \cos 2x) dy = (e^x + \cos 2x) dx$$

Now integrating gives the following:

$$\int y^{-6} dy = \int e^x + \cos 2x dx$$

Most antiderivatives here are basic, but $\int \cos 2x dx$ requires the substitution $u = 2x$ with $\frac{1}{2} dx = du$ so that

$$\int \cos 2x dx = \frac{1}{2} \int \cos u du = \frac{1}{2} \sin u = \frac{1}{2} \sin 2x.$$

After all integrations we have

$$-\frac{1}{5}y^{-5} = e^x + \frac{1}{2} \sin 2x + C.$$

Now solving for y gives

$$\begin{aligned} y^{-5} = -3e^x - \frac{3}{2} \sin 2x - 3C &\Rightarrow \frac{1}{y^5} = -3e^x - \frac{3}{2} \sin 2x - 3C \\ \Rightarrow y^5 = \frac{1}{-3e^x - \frac{3}{2} \sin 2x - 3C} \\ \Rightarrow y = \sqrt[5]{\frac{1}{-3e^x - \frac{3}{2} \sin 2x - 3C}} \end{aligned}$$

- (b) In other parts of the game *Super Mario Galaxy*, the force exerted when x units from a black hole is given by

$$F(x) = \frac{4}{x^2 + 1}.$$

Set up and evaluate the integral for the amount of work required for Mario to jump from a planet that is 10 units from the black hole to an above planet that is 15 units from the black hole.

You may write your final answer using the evaluation bar notation

The work performed by Mario is given by:

$$W = \int_{10}^{15} \frac{4}{x^2 + 1} dx = 4 \int_{10}^{15} \frac{1}{x^2 + 1} dx = (4 \arctan x) \Big|_{10}^{15}$$