Final Exam

MATH 106 - A&B Winter 2016

Version A

Name: ______________________________ **Section (A or B):** ________

Instructions:

- Answer as many of the following questions as possible.
- No cell phones or collaboration allowed. If you leave the classroom during the exam you must leave your cell phone with the instructor.
- Approved calculators are allowed.
- Additional scrap paper is available upon request.
- *Multiple choice questions:* Circle the letter corresponding to your answer. No partial credit will be awarded.
- *Short answer questions:* Show all of your work on the page of the problem. Clearly indicate your answer and the reasoning that you used to arrive at the answer. You do not have to simplify algebraic expressions.

This exam has 4 multiple choice problems and 6 short answer problems. There are a total of 100 points. Good luck!

Problem	**Possible Points**	**Points Earned**
MC	16	
5	12	
6	6	
7	12	
8	20	
9	16	
10	18	
TOTAL	**100**	

Theorems for Reference

Error bounds for approximating sums. Let $I = \int_a^b f(x)\,dx$, and let L_n, R_n, T_n, and M_n denote the left, right, trapezoid, and midpoint sums for I, each with n equal subdivisions.

- **Left- and right-rule errors.** Let K_1 be a constant such that $|f'(x)| \le K_1$ for all x in $[a,b]$. Then

$$\left|I - L_n\right| \le \frac{K_1(b-a)^2}{2n}; \qquad \left|I - R_n\right| \le \frac{K_1(b-a)^2}{2n}.$$

- **Midpoint- and trapezoid-rule errors.** Let K_2 be a constant such that $|f''(x)| \le K_2$ for all x in $[a,b]$. Then

$$\left|I - T_n\right| \le \frac{K_2(b-a)^3}{12n^2}; \qquad \left|I - M_n\right| \le \frac{K_2(b-a)^3}{24n^2}.$$

Taylor's theorem. Let $P_n(x)$ be the nth-order Taylor polynomial for $f(x)$ based at x_0. Suppose that for all x in I,

$$\left|f^{(n+1)}(x)\right| \le K_{n+1}.$$

Then

$$\left|f(x) - P_n(x)\right| \le \frac{K_{n+1}}{(n+1)!}|x - x_0|^{n+1}.$$

1. (4 points) Let $I = \int_a^b f(x)\,dx$ and suppose that $f(x)$ is decreasing and concave up on the interval $[a, b]$. Which of the following statements is true?

   A. $T_n \le I \le L_n$

   B. $I \le L_n \le R_n$

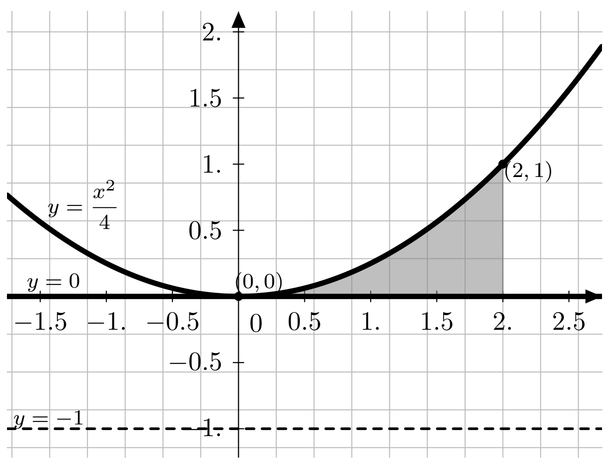
   C. $L_n \le I \le R_n$

   D. $R_n \le I \le T_n$

   E. $M_n \le I \le R_n$

   SOLUTION: Correct answer: D.

2. (4 points) Which of the following integrals describes the volume of the solid obtained by revolving the region bounded by $y = \frac{x^2}{4}$, $y = 0$, and $x = 2$ around the line $y = -1$?



   A. $\pi \int_0^2 \left(\frac{x^2}{4} - 1\right)^2 dx$

   B. $\pi \int_0^2 \left(1 + \frac{x^2}{4}\right)^2 - 1\,dx$

   C. $\pi \int_0^1 4 - \left(2\sqrt{y}\right)^2 dy$

D. $\pi \int_0^2 \left(1 + \frac{x^2}{4}\right) - 1 \, dx$

E. $\pi \int_0^1 1 - \left(1 + \frac{x^2}{4}\right)^2 \, dx$

Solution: Correct answer: B.

3. (4 points) Find $f^{(17)}(0)$ for the function $f(x)$ whose Maclaurin series is given by

$$\sum_{k=0}^{\infty} \frac{2^k x^{4k+1}}{(2k)!}.$$

A. $\dfrac{(17!)(16)}{8!}$

B. $17!$

C. $\dfrac{2^{17}}{34!}$

D. $\dfrac{16}{8!}$

E. $-\dfrac{16}{8!}$

SOLUTION: Correct answer: A.

4. (4 points) Choose the statement below that correctly identifies the behavior of the series

$$\sum_{n=2}^{\infty} (-1)^n \frac{\sqrt{n}}{n-1}.$$

A. The series diverges by the n-th term test.

B. The series is conditionally convergent by the alternating series test.

C. The series is absolutely convergent because $\lim_{n\to\infty} \dfrac{\sqrt{n}}{n-1} = 0$.

D. The series diverges by the alternating series test.

E. The series is absolutely convergent by the ratio test.

SOLUTION: Correct answer: B.

5. (12 points) Compute the following integrals. Do not use Taylor series.

(a) (6 points) $\displaystyle\int \frac{2x-1}{2x^2-2x-4}\,dx$

SOLUTION: Use u-substitution, with $u = 2x^2 - 2x - 4$ and $du = 4x - 2\,dx$.

$$\int \frac{2x-1}{2x^2-2x-4}\,dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|2x^2-2x-4| + C.$$

An alternative approach would be to use partial fraction decomposition. The partial fraction decomposition would be

$$\frac{1}{2x+2} + \frac{1/2}{x-4}, \quad \text{or} \quad \frac{1/2}{x+1} + \frac{1}{2x-4}.$$

Then the answer is

$$\frac{1}{2}\ln|2x+2| + \frac{1}{2}|x-4| + C, \quad \text{or} \quad \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|2x-4| + C.$$

By properties of the natural log, all three answers are equal.

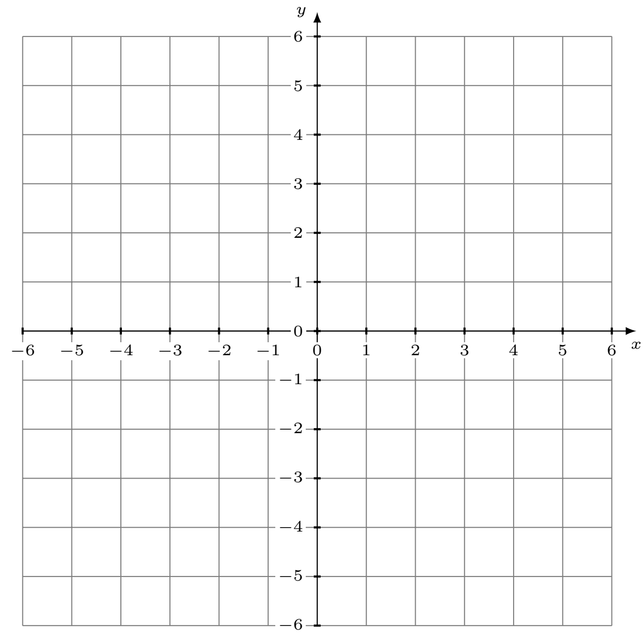
(b) (6 points) $\displaystyle\int \frac{\ln x}{x^2}\,dx$

SOLUTION: Use integration by parts, with $u = \ln x$ and $dv = \frac{1}{x^2}\,dx$. Then $du = \frac{1}{x}\,dx$ and $v = -\frac{1}{x}$. We have

$$\begin{aligned}\int \frac{\ln x}{x^2}\,dx &= -\frac{\ln x}{x} + \int \frac{1}{x^2}\,dx \\ &= -\frac{\ln x}{x} - \frac{1}{x} + C.\end{aligned}$$

6. (6 points)

(a) (2 points) Draw the region enclosed by the line $y = -2x + 4$ and the parabola $y = -x^2 + 4$ and label the points of intersection.



(b) (4 points) Set up (**but do not evaluate**) an integral with respect to x that computes the area of the enclosed region. *You do not have to simplify your answer.*

SOLUTION:

$$\int_0^2 (-x^2 + 4) - (-2x + 4)\,dx$$

7. (12 points)

(a) (6 points) Determine whether the improper integral is convergent or divergent. If the integral is convergent, evaluate it. Show all of your work for the integration.

$$\int_1^\infty \frac{2x}{(x^2+2)^2}\,dx$$

SOLUTION: We will use a u-substitution with $u = x^2 + 2$, $du = 2x\,dx$.

$$\begin{aligned}\int_1^\infty \frac{2x}{(x^2+2)^2}\,dx &= \lim_{t\to\infty} \int_1^t \frac{2x}{(x^2+2)^2}\,dx \\ &= \lim_{t\to\infty} \int_3^t \frac{du}{u^2} \\ &= \lim_{t\to\infty} -\frac{1}{u}\Big|_3^t \\ &= \lim_{t\to\infty} \left(-\frac{1}{t} + \frac{1}{3}\right) \\ &= \frac{1}{3}.\end{aligned}$$

Since the value of the limit is a finite real number, the improper integral converges and

$$\int_1^\infty \frac{2x}{(x^2+2)^2}\,dx = \frac{1}{3}.$$

(b) (6 points) Use part (a) and the integral test to determine the behavior of the series $\sum_{k=1}^{\infty} \frac{2k}{(k^2+2)^2}$. Verify all assumptions of the integral test. If the series converges, find an upper and lower bound for the value of its sum.

SOLUTION: Let $f(x) = \frac{2x}{(x^2+2)^2}$. Then $f(x) > 0$ for $x \geq 1$ (because the numerator and denominator are both positive) and $f(x)$ is continuous when $x \geq 1$ (because $f(x)$ is a rational function and $(x^2+2)^2 \neq 0$). Going further,

$$f'(x) = \frac{2(x^2+2)(-3x^2+2)}{(x^2+2)^4} < 0, \quad \text{when } x \geq 1.$$

Thus, we can apply the integral test. By part (a), the improper integral $\int_1^\infty f(x)\,dx$ converges to $\frac{1}{3}$. Therefore the series $\sum_{k=1}^{\infty} \frac{2k}{(k^2+2)^2}$ converges, and

$$\frac{1}{3} \leq \sum_{k=1}^{\infty} \frac{2k}{(k^2+2)^2} \leq \frac{2}{9} + \frac{1}{3} = \frac{5}{9}.$$

8. (20 points) Determine if each of the following series is convergent or divergent. Justify your answers.

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^3}$

SOLUTION: Use the comparison test. Note that the terms in the series are all positive. Then for $n \geq 1$, $n^2+2 > n^2$. Thus $\dfrac{n^2+2}{n^3} > \dfrac{n^2}{n^3} = \dfrac{1}{n}$. Now the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it is the harmonic series. Thus by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+2}{n^3}$ diverges.

(b) (8 points) $\displaystyle\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n}$

SOLUTION: The first few terms in this sum are

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} = -2 + \frac{4}{3} - \frac{8}{9} + \frac{16}{27} - \dots.$$

This is a geometric series with $r = -\dfrac{2}{3}$. Since $|r| < 1$, the series converges.

(c) (4 points) If possible, find the sum of the series in part (b). If this is not possible, leave this problem blank.

SOLUTION: Since the series in part (b) is a convergent geometric series, its sum will equal $\dfrac{a}{1-r}$, where a is the first term in the series and r is the common ratio. In the example above $a = -2$ and $r = -\dfrac{2}{3}$, thus

$$\sum_{n=0}^{\infty} \frac{(-2)^{n+1}}{3^n} = \frac{-2}{1-(-2/3)} = -\frac{8}{5}.$$

9. (16 points) Find the radius and interval of convergence for the following power series.

(a) (8 points) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x+1)^n}{4^n}$

SOLUTION: Use the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(n+1)(x+1)^{n+1}}{4^{n+1}} \right| \cdot \left| \frac{4^n}{n(x+1)^n} \right| \\ &= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{4^n}{4^{n+1}} \cdot \frac{|x+1|^{n+1}}{|x+1|^n} \\ &= \lim_{n\to\infty} \frac{n+1}{n} \cdot \frac{1}{4} \cdot |x+1| \\ &= \frac{|x+1|}{4}. \end{aligned}$$

Since $L = \frac{|x+1|}{4}$, the series converges when $\frac{|x+1|}{4} < 1$. In other words, it converges when $-5 < x < 3$. The series will diverge when $x < -5$ or $x > 3$. At $x = -5$, the series becomes $\sum_{n=1}^{\infty} (-1)^n n$. This series diverges by the n-th term test. At $x = 3$, the series becomes $\sum_{n=1}^{\infty} n$. This series also diverges by the n-th term test.

Thus the interval of convergence is $-5 < x < 3$ and the radius of convergence is 4.

Radius of convergence: __________ Interval of convergence: __________

(b) (8 points) $\displaystyle\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$

SOLUTION: Use the ratio test.

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{x^{2(n+1)}}{(2(n+1))!} \right| \cdot \left| \frac{(2n)!}{x^{2n}} \right| \\ &= \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \cdot \frac{|x|^{2n+2}}{|x|^{2n}} \\ &= \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} \cdot |x|^2 \\ &= 0. \end{aligned}$$

Thus $L = 0$, so the power series converges for all values of x. The radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

Radius of convergence: __________ Interval of convergence: __________

10. (18 points) For this problem, consider the function $f(x) = e^{-x^2}$.

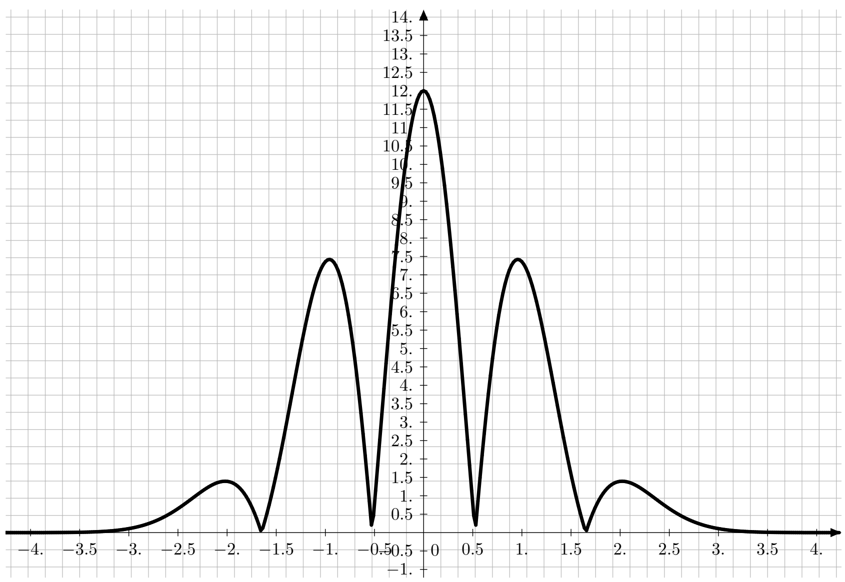
(a) (4 points) Find $P_3(x)$, the third-degree Maclaurin polynomial for $f(x)$.

SOLUTION:

$f^{(k)}(x)$	$f^{(k)}(0)$	$a_k = \dfrac{f^{(k)}(0)}{k!}$
$f(x) = e^{-x^2}$	1	1
$f'(x) = -2xe^{-x^2}$	0	0
$f''(x) = -2e^{-x^2} + 4x^2e^{-x^2}$	-2	$\dfrac{-2}{2!} = -1$
$f'''(x) = 12xe^{-x^2} - 8x^3e^{-x^2}$	0	0

$$P_3(x) = 1 + 0x - x^2 + 0x^3 = 1 - x^2$$

(b) (2 points) What does Taylor's theorem imply about the maximum error committed by P_3 over the interval $[-1, 1]$? A plot of $|f^{(4)}(x)|$ is given below.



SOLUTION: From the plot, $|f^{(4)}(x)| \leq 12$, so I will use $K_4 = 12$. For x in $[-1, 1]$, $|x - x_0| = |x - 0| \leq 1$. Then by Taylor's theorem,

$$|f(x) - P_3(x)| \leq \frac{12}{4!}|1|^4 = \frac{1}{2}.$$

(c) (6 points) Find the Maclaurin series for $f(x) = e^{-x^2}$. *Write out at least 4 non-zero terms of the series, or write in sigma notation.*

SOLUTION: First write the Maclaurin series for $f(x) = e^x$:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

Then

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots$$

(d) (6 points) Evaluate $\int e^{-x^2}\, dx$ as an infinite series. *Write out at least 4 non-zero terms of the series, or write in sigma notation.*

SOLUTION: Integrate the series in part (a),

$$\begin{aligned}
\int e^{-x^2}\, dx &= \int \left(1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \dots\right) dx \\
&= C + x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \dots \\
&= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)n!}.
\end{aligned}$$