

Math 106: Review for Final Exam, Part I - SOLUTIONS

1. **Find the following.** [See Review for Exam II for integration tips and strategies.]

(a) Let $u = x^3$, so $du = 3x^2 dx$ and $du/3 = x^2 dx$.

$$\begin{aligned} \int 12x^2 \cos(x^3) dx &= 12 \int \cos(x^3) x^2 dx \\ &= 12 \int \cos(u) \frac{du}{3} \\ &= 4 \sin(u) + C \\ &= 4 \sin(x^3) + C \end{aligned}$$

(b) We'll use integration by parts: $u = x \Rightarrow du = dx$ and $dv = e^{-3x} \Rightarrow v = \frac{e^{-3x}}{-3}$.

$$\begin{aligned} \int_0^\infty x e^{-3x} dx &= \lim_{t \rightarrow \infty} \int_0^t x e^{-3x} dx \\ &= \lim_{t \rightarrow \infty} \left[x \frac{e^{-3x}}{-3} \Big|_0^t - \int_0^t \frac{e^{-3x}}{-3} dx \right] \\ &= \lim_{t \rightarrow \infty} \left[x \frac{e^{-3x}}{-3} - \frac{e^{-3x}}{9} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{-x}{3e^{3x}} - \frac{1}{9e^{3x}} \right]_0^t \\ &= \lim_{t \rightarrow \infty} \left[\frac{-t}{3e^{3t}} - \frac{1}{9e^{3t}} \right] - \left[\frac{0}{3e^0} - \frac{1}{9e^0} \right] \\ &= (0 - 0) - (0 - 1/9) \\ &= 1/9 \end{aligned}$$

So, the integral converges (to this value).

(c) This integral is improper at $x = 4$ because the integrand has a vertical asymptote there, so we split into two integrals.

$$\begin{aligned} \int_0^6 \frac{dx}{(x-4)^2} &= \int_0^4 \frac{dx}{(x-4)^2} + \int_4^6 \frac{dx}{(x-4)^2} \\ &= \lim_{a \rightarrow 4^-} \int_0^a \frac{dx}{(x-4)^2} + \lim_{b \rightarrow 4^+} \int_b^6 \frac{dx}{(x-4)^2} \\ &= \lim_{a \rightarrow 4^-} \frac{-1}{(x-4)} \Big|_0^a + \lim_{b \rightarrow 4^+} \frac{-1}{(x-4)} \Big|_b^6 \qquad \int u^{-2} du = -u^{-1} + C \\ &= \lim_{a \rightarrow 4^-} \left[\frac{-1}{(a-4)} - \frac{-1}{(0-4)} \right] + \lim_{b \rightarrow 4^+} \left[\frac{-1}{(6-4)} - \frac{-1}{(b-4)} \right] \end{aligned}$$

Since $\lim_{a \rightarrow 4^-} \frac{-1}{(a-4)} = \infty$ and $\lim_{b \rightarrow 4^+} \frac{-1}{(b-4)} = \infty$, this integral diverges (to ∞).

(d) Partial Fractions:

Write $\frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 4}$. Now multiply both sides by $(x^2 + 1)(x - 4)$ to get

$$3x^2 + 2x - 5 = (Ax + B)(x - 4) + C(x^2 + 1).$$

Let $x = 4$. Then $51 = C(17)$, so $C = 3$.

Let $x = 0$. Then $-5 = B(-4) + 3(1)$, so $B = 2$.

Let $x = 1$. Then $0 = (A(1) + 2)(-3) + 3(2)$, so $A = 0$.

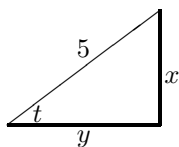
$$\begin{aligned} \int \frac{3x^2 + 2x - 5}{(x^2 + 1)(x - 4)} dx &= \int \left[\frac{2}{x^2 + 1} + \frac{3}{x - 4} \right] dx \\ &= 2 \arctan x + 3 \ln |x - 4| + D \end{aligned}$$

(e) Let $u = \sec x$, so $du = \sec x \tan x dx$.

New limits: $x = 0 \Rightarrow u = \sec 0 = 1/\cos 0 = 1$ and $x = \pi/3 \Rightarrow u = \sec(\pi/3) = 1/\cos(\pi/3) = 2$.

$$\begin{aligned} \int_0^{\pi/3} \tan^3 x \sec^5 x dx &= \int_0^{\pi/3} \tan^2 x \sec^4 x \sec x \tan x dx && \text{Break off a } \sec x \tan x. \\ &= \int_0^{\pi/3} (\sec^2 x - 1) \sec^4 x \sec x \tan x dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\ &= \int_1^2 (u^2 - 1)u^4 du && \text{Change the limits. See above.} \\ &= \int_1^2 (u^6 - u^4) du \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5} \right]_1^2 \\ &= \left[\frac{2^7}{7} - \frac{2^5}{5} \right] - \left[\frac{1^7}{7} - \frac{1^5}{5} \right] \\ &= \frac{418}{35} && \text{This is about 11.943.} \end{aligned}$$

(f) Let $x = 5 \sin t$, so $dx = 5 \cos t dt$.



$$\begin{aligned} x^2 + y^2 &= 5^2 \Rightarrow y = \sqrt{25 - x^2} \\ \sin t &= \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5) \\ \cos t &= \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5 \cos t = \sqrt{25 - x^2} \end{aligned}$$

$$\begin{aligned} \int \sqrt{25 - x^2} dx &= \int 5 \cos t \cdot 5 \cos t dt && \text{Use } dx \text{ and } \cos t \text{ from above.} \\ &= \int 25 \cos^2 t dt \\ &= 25 \int \left[\frac{1}{2} + \frac{\cos(2t)}{2} \right] dt && \text{Use } \cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2} \text{ or table \#42.} \\ &= 25 \left[\frac{t}{2} + \frac{\sin(2t)}{4} \right] + C && \text{Let } u = 2t \text{ to integrate } \cos(2t). \\ &= 25 \left[\frac{\arcsin(x/5)}{2} + \frac{2 \sin t \cos t}{4} \right] + C && \text{Use } \sin(2t) = 2 \sin t \cos t \text{ and } x \text{ from above.} \\ &= 25 \left[\frac{\arcsin(x/5)}{2} + \frac{2 \cdot \frac{x}{5} \cdot \frac{\sqrt{25 - x^2}}{5}}{4} \right] + C && \text{Use } \sin t \text{ and } \cos t \text{ from above.} \\ &= 25 \left[\frac{\arcsin(x/5)}{2} + \frac{x\sqrt{25 - x^2}}{50} \right] + C \end{aligned}$$

2. Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to $\int_{-2}^0 f(x) dx$ given the data in the table below.

x	-2	-1.5	-1	-0.5	0
$f(x)$	2	3	6	10	11

$$L_4 = (2 + 3 + 6 + 10)(0.5) = 10.5 \quad R_4 = (3 + 6 + 10 + 11)(0.5) = 15 \quad T_4 = 0.5(L_4 + R_4) = 12.75$$

We cannot compute M_4 , which would require the values of f at $x = -1.75, -1.25, -0.75$, and -0.25 . Instead, we find M_2 : $M_2 = (3 + 10)(1) = 13$.

$$\text{Finally, } S_4 = \frac{2M_2 + T_2}{3}, \text{ so we need to compute } T_2 = \frac{L_2 + R_2}{2} = \frac{(2 + 6)(1) + (6 + 11)(1)}{2} = 12.5.$$

$$\text{Thus, } S_4 = \frac{2M_2 + T_2}{3} = \frac{2(13) + 12.5}{3} = \frac{77}{6}.$$

3. If you use numerical integration to estimate $\int_a^b \ln x dx$ (where a and b are positive), how would the following be ordered from least to greatest? $L_{100}, R_{100}, M_{100}, T_{100}, \int_a^b \ln x dx$.

The integrand is increasing and concave down, so we have $L_{100} < T_{100} < \int_a^b \ln x dx < M_{100} < R_{100}$.

4. Find bounds for each of the following errors if $I = \int_0^2 e^{-5x} dx$.

$$(a) |I - R_{100}| \leq \frac{K_1(b-a)^2}{2n} = \frac{5(2-0)^2}{2(100)} = \frac{1}{10}$$

$$K_1 = \max \text{ of } |f'(x)| \text{ on } [0, 2] = \max \text{ of } 5e^{-5x} \text{ on } [0, 2] = 5 \text{ (occurs at } x = 0)$$

$$(b) |I - T_{100}| \leq \frac{K_2(b-a)^3}{12n^2} = \frac{25(2-0)^3}{12(100)^2} = \frac{1}{600}$$

$$K_2 = \max \text{ of } |f''(x)| \text{ on } [0, 2] = \max \text{ of } 25e^{-5x} \text{ on } [0, 2] = 25 \text{ (occurs at } x = 0)$$

$$(c) |I - M_{100}| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{25(2-0)^3}{24(100)^2} = \frac{1}{1200}$$

$$K_2 = \text{same as in previous part}$$

5. Write an integral equal to the area between $y = 2x + 3$ and $y = x^2 + 7x - 3$.

First, find where the curves intersect.

$$x^2 + 7x - 3 = 2x + 3$$

$$x^2 + 5x - 6 = 0$$

$$(x + 6)(x - 1) = 0$$

$$\Rightarrow x = -6, x = 1$$

Between $x = -6$ and $x = 1$, $y = 2x + 3$ is above $y = x^2 + 7x - 3$. (Plug in $x = 0$ to check.) So, the area between them is $\int_{-6}^1 [(2x + 3) - (x^2 + 7x - 3)] dx$. [This equals 343/6.]

6. Compute the arc length of $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1/2$.

$$\text{First, we find } f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}}.$$

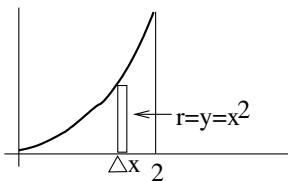
$$\text{Thus, } [f'(x)]^2 = \frac{x^2}{1 - x^2}.$$

$$\begin{aligned}
\int_a^b \sqrt{1 + [f'(x)]^2} dx &= \int_0^{1/2} \sqrt{1 + \frac{x^2}{1-x^2}} dx && \text{This is the definition of arc length.} \\
&= \int_0^{1/2} \sqrt{\frac{1-x^2}{1-x^2} + \frac{x^2}{1-x^2}} dx && \text{Get a common denominator.} \\
&= \int_0^{1/2} \sqrt{\frac{1}{1-x^2}} dx \\
&= \int_0^{1/2} \frac{\sqrt{1}}{\sqrt{1-x^2}} dx \\
&= \arcsin x \Big|_0^{1/2} \\
&= \arcsin(1/2) - \arcsin(0) \\
&= \pi/6 - 0 \\
&= \pi/6
\end{aligned}$$

7. Consider the region bounded by $y = 0$, $x = 2$, and $y = x^2$. Write an integral equal to the volume of the object created when the region is revolved about

(a) the x -axis

Slice vertically into disks.

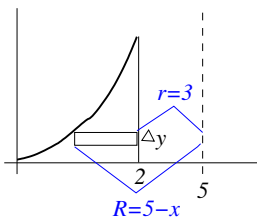


$$\begin{aligned}
\text{volume of slice} &\approx \pi r^2 \Delta x \\
&= \pi y^2 \Delta x \\
&= \pi (x^2)^2 \Delta x \\
&= \pi x^4 \Delta x
\end{aligned}$$

$$\text{total volume} = \pi \int_0^2 x^4 dx$$

(b) the line $x = 5$

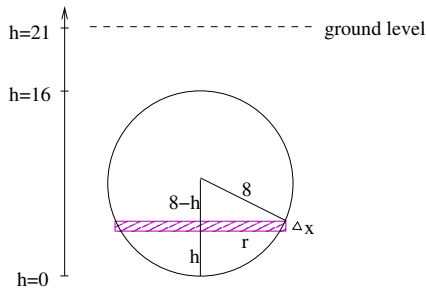
Slice horizontally into washers.



$$\begin{aligned}
\text{volume of slice} &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\
&= \pi (5-x)^2 \Delta y - \pi (3)^2 \Delta y \\
&= \pi [(5-\sqrt{y})^2 - 3^2] \Delta y
\end{aligned}$$

$$\text{total volume} = \pi \int_0^4 [(5-\sqrt{y})^2 - 3^2] dy$$

8. A spherical tank of radius 8 feet is buried 5 feet below ground and filled to a height of 11 feet with gasoline (42 pounds per cubic foot). Write an integral equal to the work done in pumping all the gasoline to ground level. [Students in the 1:10 section should omit this problem.]



$$\begin{aligned} \text{volume of slice} &\approx \pi r^2 \Delta h = \pi(16h - h^2)\Delta h \\ \text{weight of slice} &\approx 42\pi(16h - h^2)\Delta h \\ \text{work to lift slice} &\approx 42\pi(16h - h^2)\Delta h(21 - h) \\ \text{total work} &= 42\pi \int_0^{11} (16h - h^2)(21 - h) dh \end{aligned}$$

$$\begin{aligned} r^2 + (8 - h)^2 &= 8^2 \\ r^2 + 64 - 16h + h^2 &= 64 \\ r^2 &= 16h - h^2 \end{aligned}$$

9. The probability density function (pdf) of the weights of newborn toads in a certain pond is given by $f(x) = \frac{k}{(x+1)^4}$, where x is the weight (in ounces). Note that the domain is $x \geq 0$ since no toad can have a negative weight.

- (a) What must be the value of k ?

We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$\begin{aligned} \int_0^{\infty} \frac{k}{(x+1)^4} dx &= \lim_{t \rightarrow \infty} \int_0^t \frac{k}{(x+1)^4} dx \\ &= \lim_{t \rightarrow \infty} \left. \frac{k(x+1)^{-3}}{-3} \right|_0^t \\ &= \lim_{t \rightarrow \infty} \left. \frac{k}{-3(x+1)^3} \right|_0^t \\ &= \lim_{t \rightarrow \infty} \frac{k}{-3(t+1)^3} - \frac{k}{-3(0+1)^3} \\ &= 0 - \frac{k}{-3} \\ &= \frac{k}{3} \end{aligned}$$

So, we have $k/3 = 1$ or $k = 3$.

- (b) What fraction of the newborn toads weigh more than one ounce?

$$\begin{aligned} \int_1^{\infty} \frac{3}{(x+1)^4} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{3}{(x+1)^4} dx \\ &= \lim_{t \rightarrow \infty} \left. \frac{3}{-3(x+1)^3} \right|_1^t \\ &= \lim_{t \rightarrow \infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \\ &= 0 - \frac{1}{-8} \\ &= \frac{1}{8} \end{aligned}$$

Note that we could instead have computed $1 - \int_0^1 \frac{3}{(x+1)^4} dx$ and gotten the same answer.

10. Find the solution to $\frac{dy}{dx} = \frac{\cos x}{y^2}$ that passes through $(0, 2)$.

We use separation of variables.

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x}{y^2} \\ y^2 dy &= \cos x dx \\ \int y^2 dy &= \int \cos x dx \\ y^3/3 &= \sin x + C \\ y^3 &= 3 \sin x + D \\ y &= \sqrt[3]{3 \sin x + D}\end{aligned}$$

When $x = 0$, we have $y = 2$, so $2 = \sqrt[3]{3 \sin 0 + D}$, or $2 = \sqrt[3]{D}$. Thus, $D = 8$.
Therefore, the solution is $y = \sqrt[3]{3 \sin x + 8}$.