Math 105: Review for Final Exam, Part I - SOLUTIONS

1. **Consider the function $f(x) = \dfrac{3}{5-2x}$.**

   (a) **Is this function continuous on the interval $(-\infty, \infty)$? Explain.**
   No. f is discontinuous at $x = 2.5$, where f is undefined (and has a vertical asymptote).

   (b) **Compute the average rate of change of f on $[2, 2.01]$.**

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[\frac{3}{5-2(2.01)} - \frac{3}{5-2(2)}\right] \cdot \frac{1}{.01} \approx 6.122$$

   (c) **Using the limit definition of the derivative, compute $f'(x)$.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\
&= \lim_{h\to 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\
&= \lim_{h\to 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\
&= \lim_{h\to 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5-2(x+h)](5-2x)h} \\
&= \lim_{h\to 0} \frac{6h}{[5-2(x+h)](5-2x)h} \\
&= \lim_{h\to 0} \frac{6}{[5-2(x+h)](5-2x)} \\
&= \frac{6}{(5-2x)^2}
\end{aligned}$$

   (d) **Find the equation of the tangent line to f at $x = 2$.**
   We want $y = mx + b$. $m = f'(2) = \dfrac{6}{(5-2(2))^2} = 6$, so $y = 6x + b$.
   [Note that this slope agrees well with our answer from (b) above.]
   When $x = 2$, $y = f(2) = \dfrac{3}{5-2(2)} = 3$.
   Thus, $3 = 6 \cdot 2 + b$, so $b = -9$ and we have $y = 6x - 9$.

2. **Given that $f(0) = 2$, $g(0) = 3$, $f'(0) = 5$, $g'(0) = 7$, and $f'(3) = \pi$ compute the following.**

   (a) **$h'(0)$ if $h(z) = f(z)g(z)$**
   $h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$

   (b) **$j'(0)$ if $j(z) = \dfrac{f(z)}{g(z)}$**

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)^2]} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

   (c) **$k'(0)$ if $k(z) = f(g(z))$**
   $k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$

3. (a) **Find $\frac{dy}{dt}$ if $y = t^5 + 5^t + e^5 + \frac{t}{5} + \frac{5}{t} + \frac{5}{\sqrt[5]{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin 5$.**

$$\begin{aligned}\frac{dy}{dt} &= 5t^4 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5\cdot\frac{-1}{5}t^{-6/5} + \frac{1}{5t}\cdot 5 + \frac{1}{1+(5t)^2}\cdot 5 + 0 + 0 \\ &= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{6/5}} + \frac{1}{t} + \frac{5}{1+25t^2}\end{aligned}$$

(b) **Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x}\cos(7x^3)$.**

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}\cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3}\sin(7x^3)$$

(c) **Find $\frac{dy}{dz}$ if $y = \frac{e^z + e^\pi}{\tan 4 - 7z}$.**

$$\frac{dy}{dz} = \frac{e^z(\tan 4 - 7z) - (-7)(e^z + e^\pi)}{(\tan 4 - 7z)^2}$$

(d) **Find $\frac{dy}{dr}$ if $y = \tan\left(e^{r^2 \arcsin(5r)}\right)$.**

$$\frac{dy}{dr} = \sec^2\left(e^{r^2\arcsin(5r)}\right)\cdot e^{r^2\arcsin(5r)}\cdot\left[r^2\frac{1}{\sqrt{1-25r^2}}\cdot 5 + 2r\arcsin(5r)\right]$$

(e) **Find $\frac{dy}{dx}$ if $y^3 + yx^2 + x^2 = 3y^2$.** Here we use implicit differentiation.

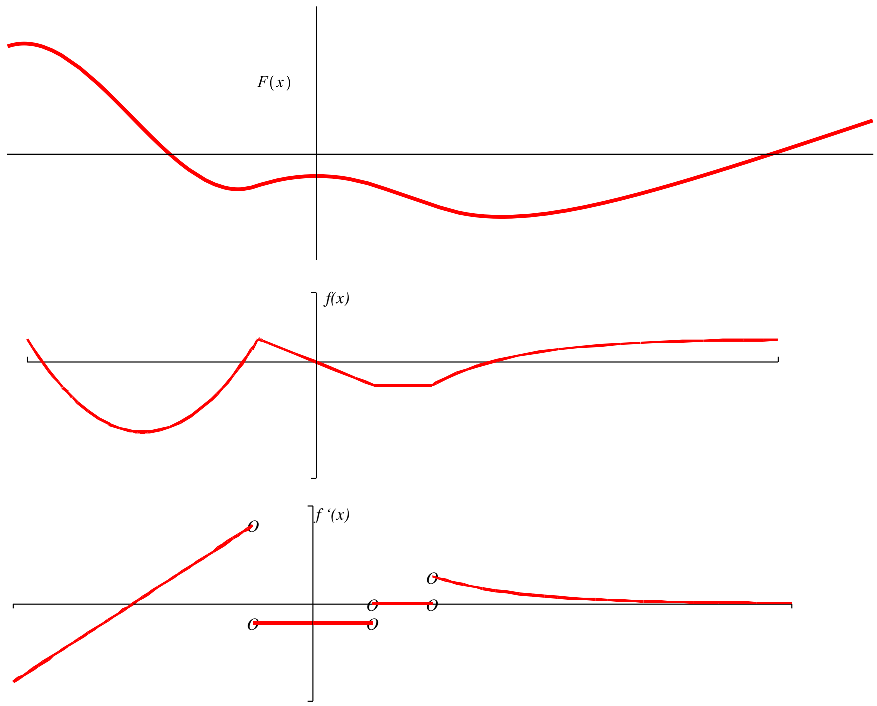
$$\begin{aligned}3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y\frac{dy}{dx} \\ 3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y\frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y}\end{aligned}$$

(f) **Find $\frac{dy}{dt}$ if $y = (1 + x^6)^{8x}$.**

Since we have x in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned}\ln y &= 8x\ln(1+x^6) \\ \frac{1}{y}\cdot\frac{dy}{dx} &= 8\cdot\ln(1+x^6) + 8x\cdot\frac{1}{1+x^6}\cdot 6x^5 \\ \frac{dy}{dx} &= \left[8\cdot\ln(1+x^6) + \frac{48x^6}{1+x^6}\right]\cdot y \\ \frac{dy}{dx} &= \left[8\cdot\ln(1+x^6) + \frac{48x^6}{1+x^6}\right]\cdot(1+x^6)^{8x}\end{aligned}$$

4. **Given the graph of f, sketch a graph of f' and a graph of F, an antiderivative of f such that $F(0) = -1$.**



Note: The concave up portion on the left side of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f, your graph of f' may be concave up/down there.

5. **Shown below is a graph of f' on its entire domain. The graph is NOT f.**

At which x-value(s) (if any)

(a) **does f have a stationary point?** k

(b) **does f have a local max?** k

(c) **does f have a local min?** none

(d) **does f' have a stationary point?** d, g, i, m

(e) **does f' have a local max?** d, i

(f) **does f' have a local min?** g, m

(g) **is f greatest?** k

(h) **is f least?** a

(i) **is f' greatest?** d

(j) **is f' least?** m

(k) **is f'' greatest?** c

(l) **is f'' least?** e

On what interval(s)* is

(a) **f increasing?** a to k

(b) **f decreasing?** k to n

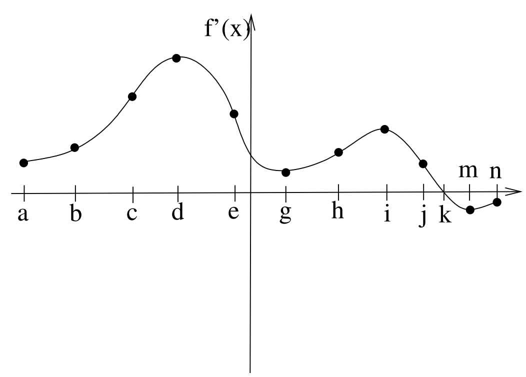
(c) **f' increasing?** a to d and g to i and m to n

(d) **f' decreasing?** d to g and i to m

(e) **f concave up?** a to d and g to i and m to n

(f) **f concave down?** d to g and i to m

* Whether to include the endpoints of these intervals will depend on your instructor's definitions.



6. **Solve the IVP $y' = e^x - \sin x + 5$ given that $y(0) = 3$.**

We antidifferentiate each side to obtain $y(x) = e^x + \cos x + 5x + C$. To find C, we let $x = 0$, meaning $3 = e^0 + \cos 0 + 5 \cdot 0 + C$, so $C = 1$ and our solution is $y(x) = e^x + \cos x + 5x + 1$.

7. **Evaluate the following limits.**

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\overset{\text{“0/0”}}{=}$ or $\overset{L'H}{=}$ or $\overset{H}{=}$ or = “0/0” or “has the form ‘$\frac{0}{0}$’ and so, by L'Hopital's Rule, is equal to” or something else. The symbol ♡ will serve the same purpose for the indeterminate form ∞/∞.

(a) $\displaystyle \lim_{x\to\infty} \frac{x^2}{\ln x} ♡ \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty$

(b) $\displaystyle \lim_{z\to 0} \frac{\sin(5z) - 5z}{z^3} ★ \lim_{z\to 0} \frac{5\cos(5z) - 5}{3z^2} ★ \lim_{z\to 0} \frac{-25\sin(5z)}{6z} ★ \lim_{z\to 0} \frac{-125\cos(12z)}{6} = -\frac{125}{6}$

(c) $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$

(d) $\displaystyle \lim_{r\to 2} \frac{r^3 - 8}{r - 2} ★ \lim_{r\to 2} \frac{3r^2}{1} = 12$

8. **Consider the function $f(x) = x^6 - 2x^3$ on the interval $[-2, 2]$.**

(a) **Find the x- and y-coordinates of any and all critical points and use calculus to classify each as a local maximum, local minimum, or neither.**

$f'(x) = 6x^5 - 6x^2$ Since $f'(x)$ never fails to exist, we just need to solve $f'(x) = 0$.

$0 = 6x^2(x^3 - 1)$

$\Rightarrow x = 0, 1$

	$-2 \le x < 0$	$0 < x < 1$	$1 < x \le 2$
f'	negative	negative	positive
f	↘	↘	↗

y-values: $f(0) = 0$, $f(1) = -1$

So, f has a local minimum at $(1, -1)$; $(0, 0)$ is not a local extremum.

(b) **Find the x- and y-coordinates of any and all global extrema and classify each as a global maximum or global minimum.**

We check the y-values at the local extrema and the endpoints.

y-values: $f(-2) = 80$, $f(1) = -1$, $f(2) = 48$

So, f has a global minimum at $(1, -1)$ and a global maximum at $(-2, 80)$.

(c) **Find the x-coordinate(s) of any and all inflection points.**

$f''(x) = 30x^4 - 12x$ Since $f''(x)$ never fails to exist, we just need to solve $f''(x) = 0$.

$0 = 6x(5x^3 - 2)$

$\Rightarrow x = 0, \sqrt[3]{0.4}$

	$x < 0$	$0 < x < \sqrt[3]{0.4}$	$\sqrt[3]{0.4} < x$
f''	positive	negative	positive
f	concave up	concave down	concave up

So, the x-values of the inflection points of f are $x = 0$ and $x = \sqrt[3]{0.4}$.