1. Let \( R \) be the region shown in the figure to the right; the curves forming its boundaries are labeled.

1A. Find the area of \( R \) by setting up and evaluating the appropriate integral(s). Show all your steps.

If you choose to set it up by \( dx \)'s,

you need two integrals:

\[
\text{Area} = \int_{-2}^{0} (y-x^2) \, dx + \int_{0}^{2} (x^2 - 4x + 4) \, dx
\]

\[
= 4x - x^3 \bigg|_{-2}^{0} + \frac{x^2}{2} - 2x + 4x \bigg|_{0}^{2}
\]

\[
= [0 - (-8 + \frac{8}{3})] + \left[ \frac{8}{3} - 8 + 8 \right] - 0
\]

\[
= 8 - \frac{8}{3} + \frac{8}{3} = 8
\]

If you choose to set it up by \( dy \)'s, you need only one integral:

\[
\int_{y=0}^{y=4} \left( 2 - \sqrt{y} \right) - \left( -\sqrt{4-y} \right) \, dy
\]

\[
\text{Note: if } y = 4 - x^2, \text{ then } x^2 = 4 - y \text{ so } x = \pm \sqrt{4-y}.
\]

But \( x \)'s are negative for the left-hand curve. So

\[
x = -\sqrt{4-y}
\]

1B. Suppose the region \( R \) is revolved around the line \( x = 3 \). Set up the integral(s) which represents the volume of the resulting solid. You do not need to evaluate your integral(s).

For \( R \), the length is \((\text{right x-coord}) - (\text{left x-coord})\):

\[
= 3 - x_r \quad \text{See the figure}
\]

\[
= 3 - \left( -\sqrt{4-y} \right) \quad \text{See the note above}
\]

\[
= 3 + \sqrt{4-y}
\]

For \( r \), the length is \((\text{right x-coord}) - (\text{left x-coord})\):

\[
= 3 - x_r \quad \text{See the figure}
\]

\[
= 3 - (2 - \sqrt{y}) = 1 + \sqrt{y}
\]

Answer:

\[
\pi \int_{y=0}^{y=4} \left( (3 + \sqrt{4-y})^2 - (1 + \sqrt{y})^2 \right) \, dy
\]
2A. Find the exact value (call it “EV”) of \( \int_0^{1.5} x \cos x^2 \, dx \) by using a substitution and the Fundamental Theorem of Calculus. Show all the steps, give the exact value and also express EV as a decimal number to as many places as your calculator displays. (For example, if you get \( \sqrt{17} \) for EV you would also list the decimal number 4.123105626. \\

\[
\int_0^{1.5} x \cos x^2 \, dx, \text{ let } u = x^2, \text{ then } du = 2x \, dx, \text{ and } \int_0^{1.5} x \cos x^2 \, dx = \frac{1}{2} \int_0^{1.5} (\cos x^2) \cdot 2x \, dx = \frac{1}{2} \int_0^{2.25} (\cos u) \, du \quad \text{"EV"} \]

\[
= \left. \frac{1}{2} \left( \sin u \right) \right|_0^{2.25} = \frac{1}{2} \left( \sin 2.25 - \sin 0 \right) = \frac{1}{2} \sin(2.25) = 0.3890365984
\]

**NOTE:** There is NO NEED to “go back to \( x \)'s” if the substitution is done right. But you can:

\[
\left. \frac{1}{2} \sin u \right|_0^{2.25} = \frac{1}{2} \sin x^2 \left. \right|_0^{1.5}
\]

\[
= \frac{1}{2} \left( \sin 2.25 - \sin 0 \right)
\]

2B. What are the limits on the integral in (2A) after the required substitution was made?

0 and 2.25 respectively (i.e. \( \int_0^{2.25} \))

2C. Find MID(100), the midpoint rule approximation of \( \int_0^{1.5} x \cos x^2 \, dx \) if 100 subdivisions are used.

(by calculator program it's) 0.38908469...

2D. What does theorem 3 of chapter 6 guarantee is the worst possible error MID(100) can give for the integral in 2A?

Two hints: (1) if \( f(x) = x \cos(x^2) \) then \( f''(x) = -6x \sin(x^2) - 4x^3 \cos(x^2) \);

(2) a nice window for plotting \( f'' \) is \([0, 1.5] \times [-10, 5]\).

**NOTE:** Choose the best \( K_2 \) to one digit after the decimal point. Give your ANSWER to eight places after the decimal point, NOT in scientific notation!

That's guarantee is: \( \frac{K_2 (1.5 - 0)^3}{24 \cdot 100^2} \), where \( K_2 \) satisfies \( |f''(x)| \leq K_2 \) for \( x \in [0, 1.5] \).

Graph of \( f'' \) and tracing it suggests \( K_2 = 8.2 \) will suffice (pulling \( K_2 \) to 2 digit as requested).

So: \( \frac{8.2 \cdot 1.5^3}{240000} \approx 0.00011531 \) (do eight places...)

2E. What is (to eight places after the decimal point) the exact error in using MID(100) as an approximation to EV?

\[
\text{Hint:} \quad \left| \text{EV} - \text{MID(100)} \right| = |0.3890365984... - 0.38908469...|
\]

\[
\approx 0.00004809
\]
3A. From memory, what is the Maclaurin series for $\cos t$? Explicitly write the series through the first five non-zero terms.

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \frac{t^8}{8!} - \cdots$$

3B. Use the correct answer to 3A to find the first five non-zero terms of the Maclaurin series for $x \cos(x^2)$.

First, a substitution of $x^2$ for $t$ gives

$$\cos x^2 = 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!} + \frac{x^{16}}{8!} - \cdots$$

Multiplying both sides by $x$ gives

$$x(\cos x^2) = x - \frac{x^5}{2!} + \frac{x^9}{4!} - \frac{x^{13}}{6!} + \frac{x^{17}}{8!} - \cdots$$

3C. Use the correct answer to 3B to find the first five non-zero terms of the Maclaurin series for $\int x \cos(x^2) \, dx$, that is, a series for an antiderivative of $x \cos(x^2)$. (You can let the constant $C$ of integration be 0).

Term-by-term integration of the series in 2B gives

$$\int x \cos(x^2) \, dx = \frac{x^2}{2} - \frac{x^6}{6 \cdot 2!} + \frac{x^{10}}{10 \cdot 4!} - \frac{x^{14}}{14 \cdot 6!} + \frac{x^{18}}{18 \cdot 8!} - \cdots$$

$$= \frac{x^2}{2} - \frac{x^6}{12} + \frac{x^{10}}{240} - \cdots$$

3D. Use just the first four terms of the answer to 3C to estimate $\int_0^{1.5} x \cos x^2 \, dx$. How does the result compare to the exact value (you found it in (2A), previous problem).

We need to evaluate the series in 3C at 1.5 (at 0, let that give 0).

One way is to carefully enter the first four terms in 3C as a function (e.g. $\Sigma$).

On your calculator, then evaluate $\Sigma(1.5)$

You'll find $[0.38709101]$, this differs from the exact value in 2A by $\approx 0.001945$.

3E. Bonus! Let $f(x) = x \cos(x^2)$. What is $f^{(9)}(0)$? Show your reasoning.

We know from 3B that $C_9 = \frac{1}{4!}$ (the coefficient of $x^9$)

We also know $C_9 = \frac{f^{(9)}(0)}{9!}$

$\therefore \quad \frac{f^{(9)}(0)}{9!} = \frac{1}{4!}$

So $f^{(9)}(0) = \frac{9!}{4!} = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 15120$
4. Find \( \int \frac{1}{x^2 \sqrt{x^2 - 9}} \, dx \) by making the substitution \( x = 3 \sec t \). Show all relevant triangles!

If \( x = 3 \sec t \) then \( dx = 3(\sec t)(\tan t) \, dt \), and
\[
\sqrt{x^2 - 9} = \sqrt{9 \sec^2 t - 9} = \sqrt{9(\sec^2 t - 1)} = 3 \tan t = 3 \sec t.
\]

So the integral becomes
\[
\int \frac{3(\sec t)(\tan t) \, dt}{9(\sec^2 t - 3 \tan t)}
\]
which cancels one of the factors in the numerator:
\[
= \int \frac{dt}{9(\sec t)} = \frac{1}{9} \int \cos t \, dt = \frac{1}{9} \sin t + C
\]

Now, if \( x = 3 \sec t \) then \( \sec t = \frac{x}{3} = \frac{\text{hyp}}{\text{adj}} \Rightarrow \) two sides of the \( \triangle \) become
\[
\begin{align*}
\frac{x}{3} \quad \text{hyp} \\
\frac{b}{3} \quad \text{adj}
\end{align*}
\]
and \( a^2 + b^2 = c^2 \) gives
\[
\begin{align*}
a^2 + b^2 &= x^2 \\
b &= \sqrt{x^2 - 9}
\end{align*}
\]

5. Find \( \int \arctan(Ax) \, dx \) using “LIATE” and integration by parts. Here \( A \) is a constant.

LIATE suggests using \( u = \arctan(Ax) \); so \( v' = A \, dx \), \( v = \frac{x}{A} \) and
\[
u = \frac{x}{A}
\]
\[
\frac{A \, du}{1 + (Ax)^2} \text{ (by the chain rule on } Ax \text{!)}
\]

so the integral is
\[
\int uv' \, dx = \int uv \, dx - \int vu' \, dx
\]

\[
= \arctan(Ax) \cdot \frac{x}{A} - \int \frac{Ax}{1 + (Ax)^2} \, dx
\]

let \( W = 1 + (Ax)^2 \) then \( dw = 2(Ax)^2 \, dx \) (chain rule)

\[
= 2A^2 \int \frac{x}{W} \, dx
\]

and so \( \int \frac{Ax}{W} \, dx \) becomes \( \frac{1}{2A} \int \frac{2A^2 x}{1 + (Ax)^2} \, dx = \frac{1}{2A} \int \frac{dw}{W} = \frac{1}{2A} \ln|W| + C
\]

Final answer:
\[
\arctan(Ax) \cdot - \frac{1}{2A} \ln(1 + (Ax)^2) + C
\]
6. Consider the power series \( s(x) = \sum_{k=1}^{\infty} \frac{9 k^4 (x-5)^k}{2k+3} \).

6A. Use the ratio test to find the interval of convergence (IOC) for this series (with the endpoints pending).

The series will be absolutely convergent (i.e., \( \sum |a_n| \) converges) when \( x \) satisfies
\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| < 1 ; \quad \text{for these } x's \quad \text{\sum} a_n \text{ will also converge (since absolutely convergent = regular convergence - see 10 in chapter 11)} \]

Now,
\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{9 (K+1)^4 (x-5)^{K+1}}{2 (K+1)+3} \cdot \frac{2 K^3}{9 K^4 (x-5)^K} \right|
\]
\[
= \lim_{k \to \infty} \left| \frac{9}{9} \cdot \frac{(K+1)^4}{K^4} \cdot \frac{2 K^3}{2 K^4} \cdot \frac{(x-5)^{K+1}}{(x-5)^K} \right|
\]
\[
= \lim_{k \to \infty} \left| \frac{1}{2} \cdot \frac{(K+1)^4}{K^4} \cdot \frac{1}{2} \cdot (x-5) \right|
\]
\[
= \frac{1}{2} |x-5| \cdot \lim_{K \to \infty} \left( \frac{K+1}{K} \right)^4 . \quad \text{As } K \to \infty, \quad \frac{K+1}{K} \to 1, \quad \text{so } \left( \frac{K+1}{K} \right)^4 \to \frac{1}{16}.
\]
\[
= \frac{1}{2} |x-5| .
\]

Again, we need \( x's \) for which \( \frac{1}{2} |x-5| < 1 \), i.e., \( |x-5| < 2 \), or, \( -2 < x-5 < 2 \); finally: \( 3 < x < 7 \).

Thus, the IOC is \((3,7)\) with the endpoints pending.

6B. Determine which, if either, endpoint belongs to the IOC, and explain your reasons.

If \( x = 7 \) the series becomes \( \sum_{k=1}^{\infty} \frac{9 K^4 (7-5)^k}{2k+3} = \sum_{k=1}^{\infty} \frac{9 K^4 \cdot 2^k}{2k+3} \)
\[
= \sum_{k=1}^{\infty} \frac{9}{8} K^4 . \quad \text{As } K \to \infty, \quad \frac{9}{8} K^4 \to 0 \text{ so by the "nth term test" the series DIVERGES.}
\]

If \( x = 3 \), the series becomes \( \sum_{k=1}^{\infty} \frac{9 K^4 (-2)^k}{8 \cdot 2^K} = \sum_{k=1}^{\infty} \frac{9}{8} K^4 (-1)^k \cdot 2^k \)
\[
= \sum_{k=1}^{\infty} \frac{9}{8} K^4 (-1)^k . \quad \text{It's an alternating series, yes, but that doesn't matter, because again } \frac{9}{8} K^4 (-1)^k \to 0 \text{ as } K \to \infty; \quad \text{so the series DIVERGES.}
\]

Thus, neither \( x = 3 \) nor \( x = 7 \) belong to the IOC. The IOC is \((3,7)\), PERIOD.
7. Consider the alternating series \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^2}} \).

7A) Explicitly write out the partial sum \( \sum_{k=2}^{5} \frac{(-1)^k}{\sqrt{k^2}} \). (No decimals, just write them with radicals)

\[
\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{16}} - \frac{1}{\sqrt{25}}
\]

7B) Explain why the series \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^2}} \) passes the alternating series test (AST), and therefore converges to some number \( M \).
(You do not need to find \( M \).

It is alternating!

As \( k \to \infty \), the terms \( \frac{1}{\sqrt{k^2}} \to 0 \) and they do so monotonically; that is, each term \( \frac{1}{\sqrt{k^2}} \) is larger than the one following it (\( \frac{1}{\sqrt{(k+1)^2}} \)).

7C) Find the partial sum \( \sum_{k=2}^{100} \frac{(-1)^k}{\sqrt{k^2}} \). Tell me values you used for B, A, and N in the LHS program.

We need \( B = 101 \) to understand why! (Make sure you don't accidentally enter \( (-1)^{A} x / x^{2/3} \)) for \( \mathbb{Z} \).

\[
\begin{align*}
A = 2 \\
P = \ 99 \\
\text{result is 0.3869528...}
\end{align*}
\]

7D) Find the smallest \( N \) for which the partial sum \( \sum_{k=2}^{N} \frac{(-1)^k}{\sqrt{k^2}} \) is within \( \epsilon = 1/1000 \) of the value \( M \) to which the series converges. Hint: remember that if an alternating series converges to \( M \), then \( |M - \sum_{k=2}^{N} (-1)^k c_k| < c_{N+1} \).

So let's find \( N \) for which \( c_{N+1} < \frac{1}{1000} \) because then \( |M - \sum_{k=2}^{N} (-1)^k c_k| < \frac{1}{1000} \),

\[
\begin{align*}
\text{Now,} \quad & c_{N+1} < \frac{1}{1000} \\
\iff & \frac{1}{\sqrt{(N+1)^2}} < \frac{1}{1000} \\
\iff & 1000 < \sqrt{(N+1)^2} = (N+1)^{3/2} \\
\iff & (1000)^{3/2} < (N+1) \\
\iff & (1000)^{3/2} - 1 < N \\
\iff & (31622.57...) - 1 < N \\
\iff & 31621.57 < N \quad \text{Choose} \ N = 31622
\end{align*}
\]

This problem continues, NEXT PAGE!
7, continued:

7E) Show all the steps involved in determining if \( \int_{2}^{\infty} x^{-2/3} \, dx \) converges, and if so to what.

By definition:

\[
\int_{2}^{\infty} x^{-2/3} \, dx = \lim_{B \to \infty} \int_{2}^{B} x^{-2/3} \, dx
\]

\[
= \lim_{B \to \infty} \left[ 3x^{1/3} \right]_{2}^{B}
\]

\[
= \lim_{B \to \infty} 3\sqrt[3]{B} - 3\sqrt[3]{2}
\]

\( \text{As } B \to \infty, \sqrt[3]{B} \text{ increases without bound,}
\]

\( \text{so } \sqrt[3]{B} \text{ does } 3\sqrt[3]{B} - 3\sqrt[3]{2}
\]

\( \therefore \int_{2}^{\infty} x^{-2/3} \, dx \text{ DIVERGES.} \)

7F) Is the series \( \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^2}} \) absolutely convergent? Explain your answer. The integral test may be useful.

We need to know if \( \sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2}} \) converges.

The integral test says the series converges \( \iff \) the corresponding integral converges,

\[
\int_{2}^{\infty} \frac{1}{\sqrt{x^2}} \, dx \text{ converges},
\]

that is, \( \iff \int_{2}^{\infty} x^{-2/3} \, dx \text{ converges.} \)

BUT in 7E we showed that \( \int_{2}^{\infty} x^{-2/3} \, dx \text{ DIVERGES,} \), and so therefore,

\[
\sum_{k=2}^{\infty} \frac{1}{\sqrt{k^2}} \text{ diverges as well. So } \sum_{k=2}^{\infty} \frac{(-1)^k}{\sqrt{k^2}} \text{ is NOT absolutely convergent.}
\]

7G) BONUS: How many days would it take your calculator to find the sum in 7D, based on how long it took to do the sum in 7C (I'll need you to tell me how long your calculator took, in seconds, to do 7C; just time it roughly).

My calculator took \( \approx 12 \) seconds to add 99 terms. Now, with \( N = \frac{3 \times 10^4}{2} = 3 \times 10^3 \), it takes about \( 79 \) times longer to do \( \frac{3 \times 10^3}{2} \). If the original calculator takes \( \approx 12 \) seconds,

this would take \( 3 \times 12 \) second \( = 3628 \) seconds.

Thus \( 3600 \) seconds is one hour. Thus, \( \approx 1.06 \) hrs

(However, it is conditionally convergent.)
8A. Explain why \( f(x) = \frac{1 - \cos(2\pi x)}{10000} \) is a probability density function on the interval \([0, 10000]\). Show all your steps in the computations of any necessary integrals.

\[
\int_0^{10000} f(x) \, dx = 1 \quad \text{as follows:}
\]

1. An antiderivative of \( \cos(2\pi x) \) is needed. Let \( u = 2\pi x \), then \( du = 2\pi \, dx \) and so

\[
\int_0^{10000} \frac{1 - \cos(2\pi x)}{10000} \, dx = \frac{1}{2\pi} \int_0^{10000} \sin u \, du = \frac{1}{2\pi} \sin(2\pi x) \bigg|_0^{10000}.
\]

This integral becomes

\[
\frac{1}{10000} \left( 10000 - 0 \right) = \frac{10000}{10000} = 1.
\]

For the intended problem: we need an antiderivative of \( \frac{1 - \cos(2\pi x)}{10000} \). Let \( u = \frac{2\pi x}{10000} \), then \( du = \frac{2\pi}{10000} \, dx \) and so

\[
\int_0^{10000} \frac{1 - \cos(2\pi x)}{10000} \, dx = \int_0^1 \frac{1 - \cos(2\pi u)}{10000} \, du.
\]

The second integral

\[
= \frac{1}{2\pi} \int_0^1 \sin u \, du
\]

so the "whole" \( F \) is

\[
\frac{1}{2\pi} \left( \sin(2\pi) - \sin(0) \right) = 0.
\]

8B. Suppose \( f \) represents the distribution of how many pictures a certain model of camera take before the camera breaks. What's the probability that if you buy such a camera, it fails before you take 2500 pictures?

The answer is represented by \( \int_0^{2500} f(x) \, dx \).

So

\[
\int_0^{2500} f(x) \, dx = \frac{1}{10000} \left( 2500 - \frac{1}{2\pi} \sin(2\pi x) \right) \bigg|_0^{2500}.
\]

\[
= \frac{1}{10000} \left( 2500 - 0 \right) = \frac{2500}{10000} = \frac{1}{4}.
\]

This is about 10% of the whole area.

In either case! (i.e., also for the intended)

\[
\text{but now, } \int_0^{2500} f(x) \, dx = \frac{1}{4} - \frac{1}{2\pi} \sin(2\pi \cdot \frac{x}{2500}) \bigg|_0^{2500}.
\]

\[
= \left( \frac{1}{4} - \frac{1}{2\pi} \sin(2\pi) \right) = 0.
\]

\[
= \left( \frac{1}{4} - \frac{1}{2\pi} \sin(\frac{\pi}{2}) \right) = 0.090045.
\]

(about 10%)
9. Set up the partial fraction decomposition for \( \int \frac{3x^2 + 2x + 1}{(x^2 - 16)(x^2 + 16)} \, dx \). You do NOT have to find the values of the A’s, B’s, C’s, and so on in your answer, and you do NOT have to do any integrals.

completely factoring the denominator gives \((x-4)(x+4)(x^2+16)\)

the deg (numerator) = 3 < deg (denominator) = 4 so no prior division is req’d,

and we expect

\[
\int \frac{3x^2 + 2x + 1}{(x^2 - 16)(x^2 + 16)} \, dx = \int \frac{A}{x-4} + \frac{B}{x+4} + \frac{Cx + D}{x^2 + 16} \, dx
\]

10. Find the value of the series 64 - 16 + 4 - 1 + 1/4 - 1/16 + \cdots

\[
= 64 \left( 1 - \frac{1}{4} + \frac{1}{16} - \frac{1}{64} + \frac{1}{256} - \frac{1}{1024} + \cdots \right)
\]

is a geometric series with \( a = 64 \) and \( r = -\frac{1}{4} \); since \( |\frac{1}{4}| < 1 \), this is

\[
= 64 \left( \frac{1}{1-\frac{1}{4}} \right) = 64 \left( \frac{1}{1+\frac{1}{4}} \right) = 64 \left( \frac{1}{\frac{5}{4}} \right) = \frac{4}{5} \cdot 64 = \frac{256}{5} = 51.2
\]

**Bonus:** Find a general formula for the sum of the first \( N \) terms in this series.

\[
\sum_{n=0}^{N-1} a \left( \frac{1 - r^{n+1}}{1-r} \right) \text{ with } a \text{ and } r \text{ as above, so:}
\]

\[
64 \left( \frac{1 - \left(\frac{1}{4}\right)^{N+1}}{1-\left(\frac{1}{4}\right)} \right) \quad \text{or} \quad \frac{256}{5} \left( 1 - \left(\frac{1}{4}\right)^{N+1} \right)
\]
11A. Either from memory or by cooking it up right now, what is the Maclaurin series, and the IOC, for $\arctan x$? (Give the first five non-zero terms).

\[
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \ldots
\]

for $x \in [-1, 1]$

11B. Either from memory or by cooking it up right now, what is the Maclaurin series, and the IOC, for $e^x$? (Give the first five non-zero terms).

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots
\]

for $x \in (-\infty, \infty)$