Math 106 Winter 2013

Final Exam (75 points)

Name: Solutions.

Show all your work to receive full credit for a problem.

Do not use the calculator integral function or any other programs on your calculator.

You may use formulas 1-18, 40-42, 50, 51 only from the table of integrals for any integral problem. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Round off your answers to four decimal places. Include units in your answers wherever possible.

There are twelve questions on five pages. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\int_a^b \sqrt{1+(f'(x))^2}\,dx$ $\qquad\qquad$ $\int u\,dv = uv - \int v\,du$

$|I-L_n| \le \frac{K_1(b-a)^2}{2n}$ $\qquad\qquad$ $|I-R_n| \le \frac{K_1(b-a)^2}{2n}$

$|I-T_n| \le \frac{K_2(b-a)^3}{12n^2}$ $\qquad\qquad$ $|I-M_n| \le \frac{K_2(b-a)^3}{24n^2}$

$$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

$$|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}\,|x-x_0|^{n+1}$$

$\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p \le 1$.

$\int_0^1 \frac{1}{x^p}\,dx$ converges for $p<1$ and diverges for $p \ge 1$.

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p>1$ and diverges for $p \le 1$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ for x in $(-\infty,\infty)$.

$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ for x in $(-\infty,\infty)$.

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ for x in $(-\infty,\infty)$.

1. (7 points) Evaluate the following integral exactly.

$$\int \frac{2x^2 - x + 7}{(3-x)(x^2+2)}\, dx$$

Partial fractions:

$$\frac{2x^2-x+7}{(3-x)(x^2+2)} = \frac{A}{3-x} + \frac{Bx+C}{x^2+2}.$$

$$2x^2-x+7 = A(x^2+2)+(Bx+C)(3-x)$$

$x=3$: $22 = 11A$. $A=2$.

$x=0$: $7 = \cancel{6}^{4} + 3C$. $C=1$.

$x=1$: $8 = 6+(B+1)(2)$ $B=0$.

So $\displaystyle\int \frac{2x^2-x+7}{(3-x)(x^2+2)}\,dx = \int \frac{2}{3-x}\,dx + \int \frac{1}{x^2+2}\,dx$.

$u = 3-x$

$du = -dx$ formula $\cancel{18}$ 13, $a=\sqrt{2}$.

$$= 2\int \frac{-du}{u} + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right).$$

$$= -2\ln|u| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.$$

$$= -2\ln|3-x| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.$$

2. (8 points) Solve the following initial value problem:

$$\frac{dy}{dx} = 3y^4 x \sin x \ , \ y(0) = 1.$$

$$\int \frac{dy}{3y^4} = \int x \sin x \, dx .$$

$$\frac{1}{3} \frac{y^{-3}}{-3} = \int x \sin x \, dx .$$

Use integration by parts for $\int x \sin x \, dx$:

$u = x$ $\qquad dv = \sin x \, dx$

$du = dx$ $\qquad v = \int \sin x \, dx = -\cos x .$

$$\int x \sin x \, dx = x(-\cos x) - \int -\cos x \, dx$$
$$= -x\cos x + \int \cos x \, dx$$
$$= -x\cos x + \sin x + C$$

So $-\frac{1}{9} \frac{1}{y^3} = -x\cos x + \sin x + C .$

$$\frac{1}{y^3} = 9x\cos x - 9\sin x - 9C .$$

$$y = \frac{1}{\sqrt[3]{9x\cos x - 9\sin x - 9C}}$$

When $x = 0$, $y = 1$.

$$1 = \frac{1}{\sqrt[3]{0 - 9C}} .$$

So $1 = \frac{1}{-9C}$. (Cube both sides).

$-9C = 1$. $\quad C = -\frac{1}{9}$.

So $y = \frac{1}{\sqrt[3]{9x\cos x - 9\sin x + 1}}$

3. (5 points) The life time (in hours) of a certain electronic component in a system is a continuous random variable, X, with the probability density function given by $f(x) = 2xe^{-x^2}$ for $x \geq 0$ (the function is zero for all other values of x). What is the probability that a randomy selected component has a life time of at least two hours?

$$\text{Probability} = \int_2^{\infty} 2x e^{-x^2} dx$$

$$= \lim_{t \to \infty} \int_2^{t} 2x e^{-x^2} dx.$$

To find $\int 2xe^{-x^2} dx$, use substitution:

$$u = -x^2 \qquad du = -2x\,dx.$$

$$\text{So } \int 2xe^{-x^2} dx = \int e^u(-du) = -e^u = -e^{-x^2}.$$

$$\lim_{t \to \infty} \int_2^{t} 2xe^{-x^2} dx = \lim_{t \to \infty} \left[-e^{-x^2}\right]_2^t$$

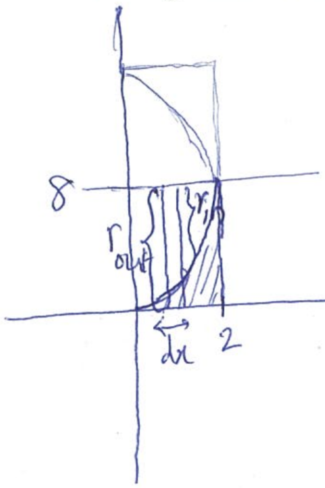
$$= \lim_{t \to \infty} \left[-e^{-t^2} + e^{-4}\right]$$

$$= \lim_{t \to \infty} \left(-\frac{1}{e^{t^2}}\right) + e^{-4}.$$

As $t \to \infty$, $e^{t^2} \to \infty$. So $-\frac{1}{e^{t^2}} \to 0$.

$$\text{So } \int_2^{\infty} 2xe^{-x^2} dx = e^{-4} = \frac{1}{e^4} \ \cancel{\text{hours}} \approx 0.0183.$$

4. (5 points) Consider the region bounded by the curve $y = x^3$, the line $x = 2$ and the x-axis. Write (but do not evaluate) an integral to find the volume of the solid that is formed when the region is rotated about the line $y = 8$.



Slice is a circle with a hole.

$r_{out} = 8$

$r_{in} = 8 - y = 8 - x^3$.

Area of slice

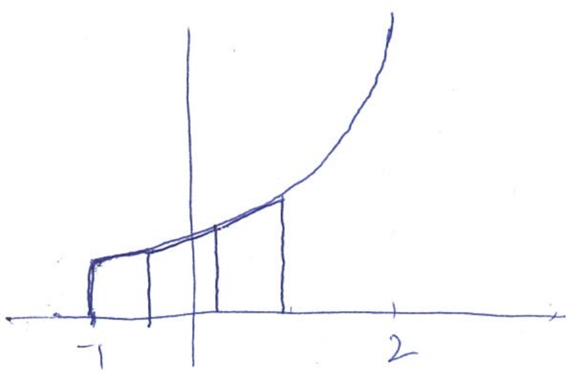
$= \pi r_{out}^2 - \pi r_{in}^2$

$= \pi (8)^2 - \pi (8 - x^3)^2$

$= 64\pi - \pi (8 - x^3)^2$.

$$\text{Volume} = \int_0^2 \left(64\pi - \pi (8 - x^3)^2\right) dx.$$

5. (4 points) Suppose a function $f(x)$ is concave up and increasing over the interval $[-1, 2]$. Does the approximation M_{50} overestimate or underestimate $\int_{-1}^{2} f(x)\, dx$? In your explanation clearly state which feature of the graph is relevant while answering this question.



The trapezoids corresponding to the midpoint rectangles are shown in the figure.

Since the graph is concave up, these trapezoids underestimate $\int_{-1}^{2} f(x)\, dx$.

Shape of graph of f(x).

So M_{50} underestimates $\int_{-1}^{2} f(x)\, dx$.

6. (7 points) Use comparisons to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=2}^{\infty} \frac{2n+3}{4n^3+5n}.$$

$4n^3+5n \geqslant 4n^3$ for all $n \geqslant 2$.

$$\frac{1}{4n^3+5n} \leq \frac{1}{4n^3}$$

$$\frac{2n+3}{4n^3+5n} < \frac{5n}{4n^3} = \frac{5}{4}\cdot\frac{1}{n^2} \text{ for all } n \geqslant 2.$$

$\sum_{n=2}^{\infty} \frac{5}{4}\cdot\frac{1}{n^2}$ converges (p- series with $p=2>1$).

So $\sum_{n=2}^{\infty} \frac{2n+3}{4n^3+5n}$ converges by comparison test.

and $$\sum_{n=2}^{\infty} \frac{2n+3}{4n^3+5n} \leq \sum_{n=2}^{\infty} \frac{5}{4}\cdot\frac{1}{n^2} \leq \frac{5}{4}\cdot\frac{1}{4} + \int_2^{\infty} \frac{5}{4}\frac{1}{x^2}\,dx.$$

$$\frac{5}{4}\int_2^{\infty} \frac{1}{x^2}\,dx = \frac{5}{4}\cdot\lim_{t\to\infty}\int_2^t \frac{1}{x^2}\,dx = \frac{5}{4}\cdot\lim_{t\to\infty}\left[\frac{x^{-1}}{-1}\right]_2^t = \frac{5}{4}\left[0+\frac{1}{2}\right] = \frac{5}{8}.$$

$$\text{So } \sum_{n=2}^{\infty} \frac{2n+3}{4n^3+5n} \leq \frac{5}{16}+\frac{5}{8} = \frac{15}{16}.$$

7. (3 points) Find the exact sum of the series $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \ldots$.

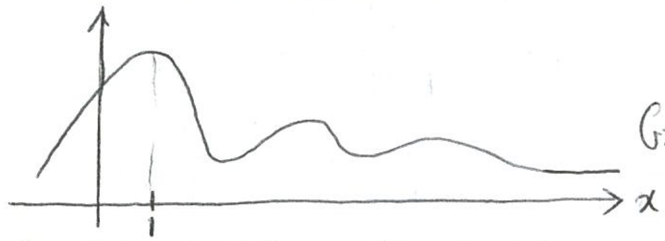
Ratio of successive terms is $-\frac{1}{5}$.

$$\frac{\frac{-3}{5}}{3} = -\frac{1}{5}, \quad \frac{\frac{3}{25}}{\frac{-3}{5}} = -\frac{1}{5}, \quad \frac{\frac{-3}{125}}{\frac{3}{25}} = -\frac{1}{5}.$$

So this is a geometric series with common ratio $r = -\frac{1}{5}$. ~~So~~

$$\text{So sum} = \frac{3}{1-\left(-\frac{1}{5}\right)} = \frac{3}{\frac{6}{5}} = \frac{5}{2} = 2.5$$

8. (6 points) The graph of a function $f(x)$ is given below. Consider the series $\sum_{n=1}^{\infty} f(n)$. Use this information to answer the questions that follow.



Graph of $f(x)$.

(a) Does the nth term test tell us anything about the convergence of the series? Explain.

From the graph, we see that $\lim_{n \to \infty} f(n) = 0$.

So the nth term test does not tell us anything about the convergence of the series.

(b) Assume that $\int_1^{\infty} f(x)\, dx$ converges. Does the integral test tell us anything about the convergence of the series? Explain.

Since $f(x)$ is not decreasing over $[1, \infty)$, the integral test does not tell us anything about the convergence of the series if we apply it over the interval $[1, \infty)$.

9. (4 points) Determine the convergence of the series $\sum_{n=2}^{\infty} n! \ln n$. If the series converges, find an upper bound on the sum of the series.

$\lim_{n \to \infty} n! \ln n = \infty$. As $n \to \infty$, $n! \to \infty$ and $\ln n \to \infty$.

So by the nth term test, the series diverges.

10. (9 points) Find the interval and radius of convergence of the series $\sum_{k=1}^{\infty} \frac{x^k}{k^2\,3^k}$.

$$a_k = \frac{x^k}{k^2 3^k}, \quad a_{k+1} = \frac{x^{k+1}}{(k+1)^2 3^{k+1}}$$

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^{k+1}}{(k+1)^2 \cdot 3^{k+1}} \times \frac{k^2 \cdot 3^k}{|x|^k} = \frac{1}{3}\frac{k^2}{(k+1)^2}|x|.$$

$$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{1}{3}\frac{k^2}{k^2+2k+1}|x| = \frac{|x|}{3} \lim_{k\to\infty} \frac{k^2}{k^2+2k+1} = \frac{|x|}{3}.$$

(Since $\lim_{k\to\infty} \frac{k^2}{k^2+2k+1} = \lim_{k\to\infty} \frac{2k}{2k+2} = \lim_{k\to\infty} \frac{2}{2} = 1$ by L'Hôpital's Rule.)

By ratio test, series converges ~~if and on~~ if $\frac{|x|}{3} < 1$ and diverges if $\frac{|x|}{3} > 1$.

$\frac{|x|}{3} < 1$ gives $|x| < 3$ i.e $-3 < x < 3$.

So we know series converges for $-3 < x < 3$ and diverges outside this interval.

Check convergence at endpoints separately:

$x=3$: Series $= \sum_{k=1}^{\infty} \frac{3^k}{k^2 3^k} = \sum_{k=1}^{\infty} \frac{1}{k^2}$ converges (p-series with $p=2>1$)

$x=-3$: Series $= \sum_{k=1}^{\infty} \frac{(-3)^k}{k^2 3^k} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$.

Use alternating series test: $a_k = \frac{(-1)^k}{k^2}$, $c_k = |a_k| = \frac{1}{k^2}$.

$\lim_{k\to\infty} c_k = 0$. $\frac{1}{(k+1)^2} < \frac{1}{k^2}$ i.e $c_{k+1} < c_k$ for all $k \geq 1$.

So by alternating series test, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges.

Hence interval of convergence $= [-3, 3]$. Radius of convergence is 3.

11. (9 points) Let $f(x) = x\cos(5x^3)$. Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for f.

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

$$\cos(5x^3) = 1 - \frac{(5x^3)^2}{2!} + \frac{(5x^3)^4}{4!} - \frac{(5x^3)^6}{6!} + \cdots = \sum_{k=0}^{\infty} \frac{(-1)^k (5x^3)^{2k}}{(2k)!}$$

$$x\cos(5x^3) = x - \frac{5^2 x^7}{2!} + \frac{5^4 x^{\cancel{\ }13}}{4!} - \frac{5^6 \cdot x^{\cancel{18}19}}{6!} + \cdots$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 5^{2k} x^{6k+1}}{(2k)!}$$

(b) Use the series in part (a) to find $f^{(55)}(0)$.

Coefficient of x^{55} in the series in part (a):

$6k+1 = 55 \quad 6k = 54 \quad k = 9.$

So coefficient is $\dfrac{(-1)^9 5^{18}}{(18)!}$.

By definition of Maclaurin series, coefficient is $\dfrac{f^{(55)}(0)}{(55)!}$

$$\frac{f^{(55)}(0)}{(55)!} = \frac{-5^{18}}{(18)!} \qquad \text{So } f^{(55)}(0) = \frac{-5^{18} \cdot (55)!}{(18)!}$$

(c) Use the series in part (a) to write the first four non-zero terms of $\int x\cos(5x^3)\,dx$.

$$x\cos(5x^3) = x - \frac{5^2 x^7}{2!} + \frac{5^4 x^{13}}{4!} - \frac{5^6 x^{19}}{6!} + \cdots$$

$$\int x\cos(5x^3)\,dx = \frac{x^2}{2} - \frac{5^2}{2!}\frac{x^8}{8} + \frac{5^4}{4!}\frac{x^{14}}{14} - \frac{5^6}{6!}\frac{x^{20}}{20} + \cdots$$

12. (8 points) Let $f(x) = \ln x$.

(a) Find the first four non-zero terms of the Taylor series for f based at $x_0 = 1$. Then write the series using the summation notation.

$f(x) = \ln x$ $\qquad$ $f(1) = 0$

$f'(x) = \frac{1}{x} = x^{-1}$ $\qquad$ $f'(1) = 1$

$f''(x) = -1 \cdot x^{-2}$ $\qquad$ $f''(1) = -1$

$f'''(x) = 2x^{-3} = 2!\,x^{-3}$ $\qquad$ $f'''(1) = 2!$

$f^{(4)}(x) = -6x^{-4} = -3!\,x^{-4}$ $\qquad$ $f^{(4)}(1) = -3!$ $\qquad$ We notice a pattern.

So Taylor series for f is

$$f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots$$

$$= 0 + 1\cdot(x-1) + (-1)(x-1)^2 + \frac{2!}{3!}(x-1)^3 + \frac{(-3!)}{4!}(x-1)^4 + \cdots$$

$$= (x-1) - \frac{(x-1)^2}{2} + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \cdots = \sum_{k=1}^{\infty} \frac{(x-1)^k(-1)^{k+1}}{k}.$$

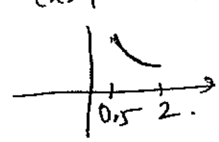
(b) Write the second order Taylor polynomial for f based at $x_0 = 1$ and use Taylor's theorem to find an upper bound on the error in estimating $\ln(1.5)$ with this Taylor polynomial. To find the best possible K_{n+1}, use the interval $[0.5, 2]$.

From the Taylor series for f in part (a), we see that $P_2(x) = (x-1) - \frac{(x-1)^2}{2}$.

By Taylor's theorem,

$$|P_2(1.5) - f(1.5)| \le \underbrace{\frac{K_3}{3!}|1.5-1|^3}_{\text{upper bound on the error}}.$$

Find K_3: $f'''(x) = \frac{2}{x^3}$. $\quad K_3 = \max|f'''(x)|$ over $[0.5, 2]$.



$K_3 = f'''(0.5) = \frac{2}{(0.5)^3} = 16.$

So $|P_2(1.5) - f(1.5)| \le \frac{16}{6}(0.5)^3 = \frac{16}{6}\cdot\frac{1}{8} = \frac{1}{3}$