Math 106 Winter 2013

Final Exam (75 points)

Name: ____

Show all your work to receive full credit for a problem.

Do not use the calculator integral function or any other programs on your calculator.

You may use formulas 1-18, 40-42, 50, 51 only from the table of integrals for any integral problem. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Round off your answers to four decimal places. Include units in your answers wherever possible.

There are twelve questions on five pages. Questions are printed on both sides of a page.

You may use any of the following facts:

Arclength$=\displaystyle\int_a^b \sqrt{1+(f'(x))^2}\,dx$ $\qquad$ $\displaystyle\int u\,dv = uv - \int v\,du$

$|I - L_n| \le \dfrac{K_1(b-a)^2}{2n}$ $\qquad$ $|I - R_n| \le \dfrac{K_1(b-a)^2}{2n}$

$|I - T_n| \le \dfrac{K_2(b-a)^3}{12n^2}$ $\qquad$ $|I - M_n| \le \dfrac{K_2(b-a)^3}{24n^2}$

$$T(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \cdots$$

$$|f(x) - P_n(x)| \le \frac{K_{n+1}}{(n+1)!}\,|x-x_0|^{n+1}$$

$\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges for $p>1$ and diverges for $p\le 1$.

$\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges for $p<1$ and diverges for $p\ge 1$.

$\displaystyle\sum_{n=1}^\infty \frac{1}{n^p}$ converges for $p>1$ and diverges for $p\le 1$.

$\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$ for x in $(-\infty,\infty)$.

$\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots$ for x in $(-\infty,\infty)$.

$e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \dfrac{x^4}{4!} + \cdots$ for x in $(-\infty,\infty)$.

1. (7 points) Evaluate the following integral exactly.

$$\int \frac{2x^2 - x + 7}{(3 - x)(x^2 + 2)}\, dx$$

2. (8 points) Solve the following initial value problem:

$$\frac{dy}{dx} = 3y^4 x \sin x \ , \ \ y(0) = 1.$$

3. (5 points) The life time (in hours) of a certain electronic component in a system is a continuous random variable, X, with the probability density function given by $f(x) = 2xe^{-x^2}$ for $x \geq 0$ (the function is zero for all other values of x). What is the probability that a randomy selected component has a life time of at least two hours?

4. (5 points) Consider the region bounded by the curve $y = x^3$, the line $x = 2$ and the x-axis. Write (but do not evaluate) an integral to find the volume of the solid that is formed when the region is rotated about the line $y = 8$.

5. (4 points) Suppose a function $f(x)$ is concave up and increasing over the interval $[-1, 2]$. Does the approximation M_{50} overestimate or underestimate $\int_{-1}^{2} f(x)\, dx$? In your explanation clearly state which feature of the graph is relevant while answering this question.

6. (7 points) Use comparisons to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{n=2}^{\infty} \frac{2n+3}{4n^3+5n}\ .$$

7. (3 points) Find the exact sum of the series $3 - \frac{3}{5} + \frac{3}{25} - \frac{3}{125} + \ldots$.

8. (6 points) The graph of a function $f(x)$ is given below. Consider the series $\sum_{n=1}^{\infty} f(n)$. Use this information to answer the questions that follow.

(a) Does the nth term test tell us anything about the convergence of the series? Explain.

(b) Assume that $\int_1^{\infty} f(x)\, dx$ converges. Does the integral test tell us anything about the convergence of the series? Explain.

9. (4 points) Determine the convergence of the series $\sum_{n=2}^{\infty} n! \ln n$. If the series converges, find an upper bound on the sum of the series.

10. (9 points) Find the interval and radius of convergence of the series $\sum_{k=1}^{\infty} \frac{x^k}{k^2\, 3^k}$.

11. (9 points) Let $f(x) = x\cos(5x^3)$. Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for f.

(b) Use the series in part (a) to find $f^{(55)}(0)$.

(c) Use the series in part (a) to write the first four non-zero terms of $\int x\cos(5x^3)\,dx$.

12. (8 points) Let $f(x) = \ln x$.

(a) Find the first four non-zero terms of the Taylor series for f based at $x_0 = 1$. Then write the series using the summation notation.

(b) Write the second order Taylor polynomial for f based at $x_0 = 1$ and use Taylor's theorem to find an upper bound on the error in estimating $\ln(1.5)$ with this Taylor polynomial. To find the best possible K_{n+1}, use the interval $[0.5, 2]$.