**Calculus I**
**MATH 105**

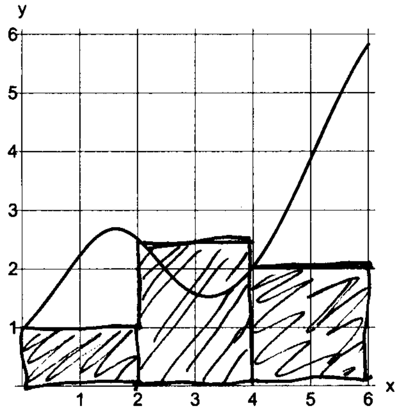
**FINAL EXAM**
**April 10, 2012**

Name: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. EXPLAIN ALL ANSWERS COMPLETELY. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

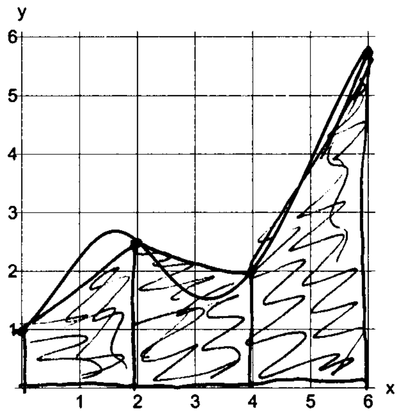
1.) (10 pts.) Use the given graphs to compute $L_3$ and $T_3$. Simplify your answers.

a.) (5 pts.) On the graph below, sketch in and compute $L_3$.




$L_3 = 2\cdot 1 + 2\cdot 2.5 + 2\cdot 2$
$= 2 + 5 + 4$
$= 11$

b.) (5 pts.) On the graph below, sketch in and compute $T_3$.




$T_3 = \frac{1}{2}(1+2.5)(2) + \frac{1}{2}(2.5+2)(2)$
$+ \frac{1}{2}(2+5.8)(2)$
$= 3.5 + 4.5 + 7.8$
$= 15.8$

2.) (15 pts.) This problem combines our three big calculus concepts: the integral, the derivative, and the limit.

a.) (5 pts.) Use the Fundamental Theorem of Calculus to compute $\int_{\frac{1}{2}}^{1}(x^{-3}-8)\,dx$.

$$= \frac{1}{-2}x^{-2} - 8x \Big|_{\frac{1}{2}}^{1}$$

$$= \frac{1}{-2x^2} - 8x \Big|_{\frac{1}{2}}^{1}$$

$$= \left(\frac{1}{-2(1)^2} - 8(1)\right) - \left(\frac{1}{-2(\frac{1}{2})^2} - 8\left(\frac{1}{2}\right)\right)$$

$$= \left(-\frac{1}{2} - 8\right) - (-2 - 4)$$

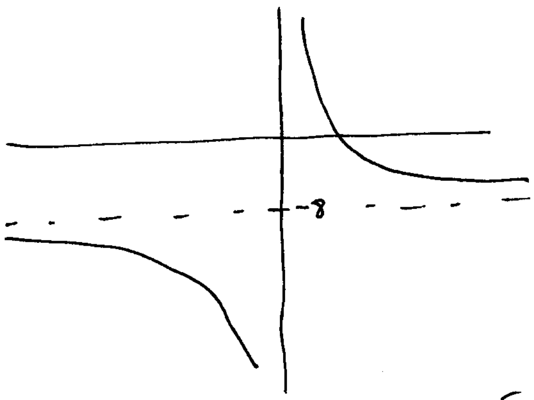
$$= -8.5 + 6$$

$$= \boxed{-2.5}$$

b.) (5 pts.) Compute the derivative of $y = x^{-3} - 8$.

$$y' = -3x^{-4}$$

c.) (5 pts.) Compute the limit $\lim_{x\to 0}(x^{-3} - 8)$. Be sure to demonstrate how you are computing the limit.



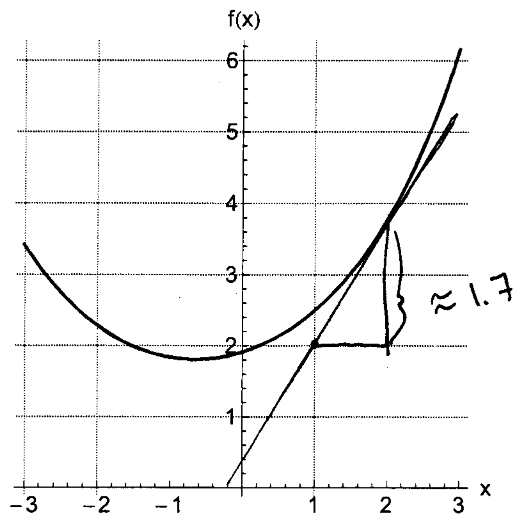

$$\lim_{x\to 0^-}(x^{-3}-8) = -\infty$$

$$\lim_{x\to 0^+}(x^{-3}-8) = +\infty$$

One-sided limits do not match.

So: $\lim_{x\to 0}(x^{-3}-8)$ D.N.E.

3.) (15 pts.) The equation $f(x) = \frac{\pi}{e} + 2^{x-1} + \frac{x^2+2}{x+8}$ is shown in the graph below.




a.) (5 pts.) Use the graph to *visually approximate* $f'(2)$.

$$\frac{\Delta y}{\Delta x} \approx \frac{1.7}{1} = \boxed{1.7}$$

b.) (5 pts.) Use the given equation to compute $f'(x)$.

$$f'(x) = 0 + \ln 2 \cdot 2^{x-1} + \frac{(x+8)(2x) - (x^2+2)(1)}{(x+8)^2}$$

$$f'(x) = \ln 2 \cdot 2^{x-1} + \frac{x^2+16x-2}{(x+8)^2}$$

c.) (5 pts.) Use your result from part (b) to compute an exact value for $f'(2)$. Compare the decimal approximation of your exact answer with your answer from part (a): how close were you?

$$f'(2) = \ln 2 \cdot 2^{2-1} + \frac{4+32-2}{10^2}$$

$$= \ln 2 \cdot 2 + \frac{34}{100}$$

$\approx 1.73$ , which is <u>very</u> close to my part (a) estimate

4.) (15 pts.) The following are two examples of techniques we have learned that require multiple steps.

a.) (7 pts.) Use L'Hôpital's Rule to compute $\lim_{x\to 0} \frac{1-\cos(3x)}{8x}$. Be sure to confirm why you *can* use L'Hôpital's Rule.

As $x \to 0$, $\cos(3x) \to 1$, so $1-\cos(3x) \to 0$

and $8x \to 0$

So: $\frac{0}{0}$ and our limit has the same value as:

$$\lim_{x\to 0} \frac{\sin(3x)\cdot 3}{8} = \frac{0}{8} = \boxed{0}$$

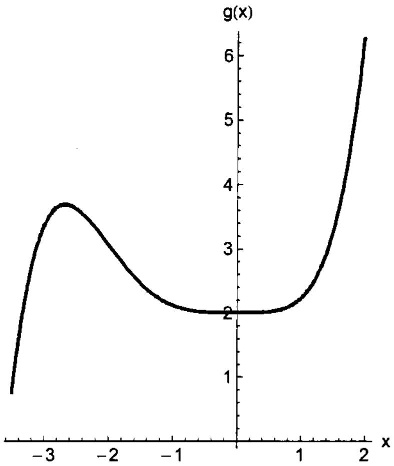
c.) (8 pts.) Use implicit differentiation to compute $\frac{dy}{dx}$ for the equation $2(x+y)^{\frac{1}{3}} = y$. Be sure to solve for $\frac{dy}{dx}$.

$$2 \cdot \frac{1}{3}(x+y)^{-\frac{2}{3}} \cdot \left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\frac{2}{3}(x+y)^{-\frac{2}{3}} + \frac{2}{3}(x+y)^{-\frac{2}{3}} \cdot \frac{dy}{dx} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\frac{2}{3}(x+y)^{-2/3}}{1 - \frac{2}{3}(x+y)^{-2/3}}$$

5.) (15 pts.) The function $g(x) = \frac{1}{20}x^5 + \frac{1}{6}x^4 + 2$ is shown in the graph below. You may choose to answer part (a) first, or wait and use your results from parts (b) and (c).




a.) (5 pts.) State the $x$-intervals for which the following hold. You can restrict $g(x)$ to the $x$-values $[-3.5, 2]$, as shown in the graph.

i.) $g(x)$ is increasing $[-3.5, -\frac{8}{3})$ , $(0, 2]$

ii.) $g(x)$ is decreasing $(-\frac{8}{3}, 0)$

iii.) $g(x)$ is concave up $(-2, 2)$

iv.) $g(x)$ is concave down $[-3.5, -2)$

b.) (5 pts.) Use calculus to show how you know the exact $x$-values at which stationary points occur.

$$g'(x) = \frac{1}{4}x^4 + \frac{2}{3}x^3 = 0$$

$$x^3\left(\frac{1}{4}x + \frac{2}{3}\right) = 0$$

$\boxed{x = 0}$ or $\frac{1}{4}x = -\frac{2}{3}$

$\rightarrow \boxed{x = -\frac{8}{3}}$

c.) (5 pts.) Use calculus to find any inflection points and confirm that they are inflection points.

$$g''(x) = x^3 + 2x^2 = 0$$

$$x^2(x+2) = 0$$

$$x = 0 \text{ or } -2$$

Confirm

no concavity change

| $g''(x)$ | − | 0 | + | 0 | + |
|---|---|---|---|---|---|
| $x$ | −3 | −2 | −1 | 0 | 1 |

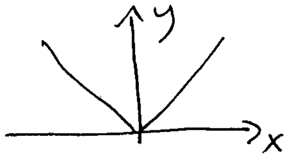
Plug in to $g''(x)$

So: only $\boxed{x = -2}$ is an inflection point.

6.) (15 pts.) The following all relate to theorems we have discussed this semester.

a.) (5 pts.) The theorem we have nicknamed the "Trick Question Theorem" says *either* "if a function is continuous, then it must be differentiable" *or* "if a function is differentiable, then it must be continuous". Which of these is correct, and what example reminds us that the other one is not correct?

Example:




$y = |x|$ is continuous everywhere, but not differentiable at $x = 0$.

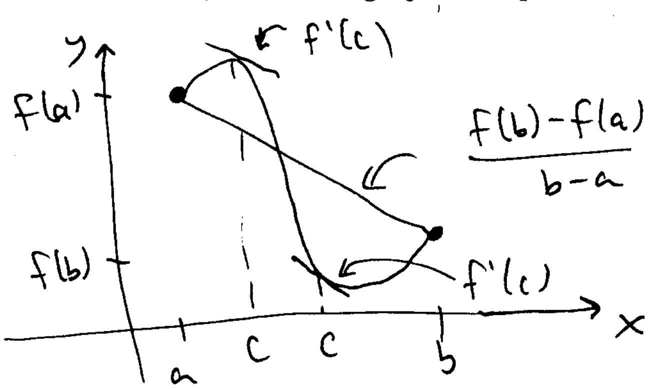
b.) (5 pts.) State the hypotheses of the Mean Value Theorem.

- $f$ is continuous on $[a, b]$
- $f$ is differentiable on ~~[a, b]~~ $(a, b)$

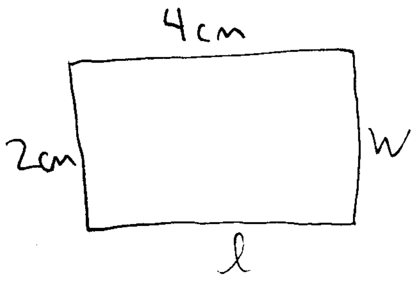
c.) (5 pts.) State the conclusion of the Mean Value Theorem, and use a graph to explain visually what the formula means.

There exists $c \in (a, b)$ for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



7.) (15 pts.) A rectangle initially has dimensions 2 cm by 4 cm. All sides begin increasing in length at a rate of 1 cm/s. At what rate is the area of the rectangle increasing after 20 s?




$$A = l w$$

$$\frac{dA}{dt} = w \cdot \frac{dl}{dt} + l \cdot \frac{dw}{dt}$$

At 20 s, $l = 4+20 = 24$ cm

$w = 2+20 = 22$ cm

$$\frac{dA}{dt} = 22 \cdot 1 + 24 \cdot 1 = 46 \ \frac{cm^2}{s}$$

**BONUS:** (5 pts.) Many of you have already done this, but in case you have not yet: by 4:00pm today (Tuesday, April 10) email me a photo of yourself at the Mount David Summit.