

# Math 106 Winter 2014

## Final Exam (75 points)

Name: \_\_\_\_\_

Show all your work to receive full credit for a problem and keep your written answers brief and clear. Points will be taken off if you do not show how you arrived at your answer, even if the final answer is correct.

Do not use the calculator integral function. Whenever possible, find the exact values of integrals by finding antiderivatives or using the table of integrals. When you use a formula from the table of integrals, mention the formula number and the value(s) of any constant(s) that you may need.

Give exact answers. If needed, round off your answers to four decimal places.

There are eleven questions on five pages. Questions are printed on both sides of a page.

You may use any of the following facts:

$$\text{Arclength} = \int_a^b \sqrt{1 + (f'(x))^2} dx \qquad \int u dv = uv - \int v du$$

$$|I - L_n| \leq \frac{K_1(b-a)^2}{2n} \qquad |I - R_n| \leq \frac{K_1(b-a)^2}{2n}$$

$$|I - T_n| \leq \frac{K_2(b-a)^3}{12n^2} \qquad |I - M_n| \leq \frac{K_2(b-a)^3}{24n^2}$$

$$T(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots$$

$$|f(x) - P_n(x)| \leq \frac{K_{n+1}}{(n+1)!} |x - x_0|^{n+1}$$

$$\int_1^\infty \frac{1}{x^p} dx \text{ converges for } p > 1 \text{ and diverges for } p \leq 1.$$

$$\sum_{n=1}^\infty \frac{1}{n^p} \text{ converges for } p > 1 \text{ and diverges for } p \leq 1.$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \text{ for } x \text{ in } (-\infty, \infty).$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \text{ for } x \text{ in } (-\infty, \infty).$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \text{ for } x \text{ in } [-1, 1].$$

1. (6 points) Suppose a function  $f$  is such that  $-3 \leq f''(x) \leq 2$  for all  $x$  in  $[1, 5]$ .

Let  $I = \int_1^5 f(x) dx$ . Use this information to answer the following questions.

(a) If possible, order the quantities  $I, T_{20}$ , and  $M_{20}$  from least to greatest, where  $T_{20}$  is the trapezoid approximation of  $I$  with 20 subintervals and  $M_{20}$  is the midpoint approximation of  $I$  with 20 subintervals. If it is not possible to order them with the given information, explain why.

(b) What is the least value of  $n$  which guarantees that a trapezoid sum approximation  $T_n$  approximates  $I$  within  $\pm 0.01$ ? Justify your answer.

2. (6 points) Solve the following initial value problem. (You may use formulas 1-18 only from the table of integrals for this problem.)

$$\frac{dy}{\cos^2 x} = y^2 \sin^3 x \, dx, \quad y(\pi/2) = 2.$$

3. (8 points) Find the exact length of the curve  $y = 2x^2$  from  $x = 0$  to  $x = 1$ . (You may use formulas 1-18, 39-42, 50, 51 only from the table of integrals for this problem.)

4. (5 points) Sketch the region bounded by the curve  $y = e^x$ , the line  $x = 1$  and the line  $y = 1$ . Write (but do not evaluate) an integral to find the volume of the solid that is formed when this region is rotated about the  $y$ -axis.

5. (5 points) Use comparison to determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series.

$$\sum_{k=0}^{\infty} \frac{7}{10 + 3^k} .$$

6. (9 points) Determine the convergence of the following series. If the series converges, find an upper bound on the sum of the series. (If you decide to use the integral test, you may use formulas 1-18 only from the table of integrals for this problem.)

$$\sum_{k=2}^{\infty} \frac{\ln k}{k^3} .$$

7. (7 points) Does the series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{5n-2}}$  converge absolutely or conditionally? Explain. (If you decide to use the integral test, you may use formulas 1-18 only from the table of integrals for this problem.)

8. (6 points) Suppose the series  $\sum_{n=0}^{\infty} a_n$  converges and its fifth tail,  $R_5 = 0.1$ . Use this information to answer the questions that follow.

(a) Does the sequence of partial sums  $S_n$  converge? Explain.

(b) If you use  $S_5$  to estimate the sum of the series, then what (if anything) can you say about the absolute error in the approximation?

9. (6 points) For each of the following series, find the exact sum if the series converges or explain why it diverges.

(a)  $-1 + \frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!} - \frac{1}{9!} + \cdots$

(b)  $1 + 1.1 + 1.01 + 1.001 + 1.0001 + \cdots$



10. (7 points) Find the interval of convergence of the series  $\sum_{k=3}^{\infty} \frac{4^k (x-2)^k}{(2k)!}$ .

11. (10 points) Let  $f(x) = \arctan(x^2)$ . Use this function to answer the following questions.

(a) Use a known power series to write the first four non-zero terms of the power series representation for  $f(x)$ . Also write this entire series using sigma notation.

(b) Find  $f^{(102)}(0)$ .

(c) Use the series in part(a) to write the third-order Taylor polynomial of  $f(x)$  based at 0.