Name: Solutions

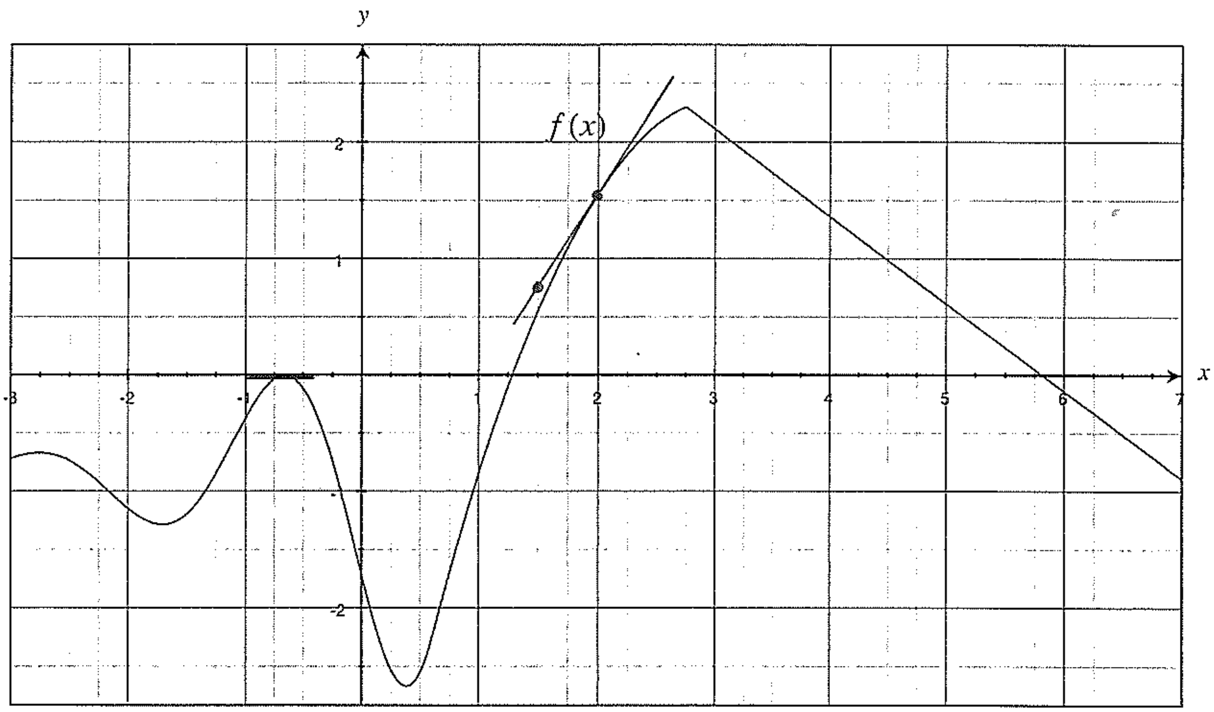
Math 105: Winter 2013
Final Exam

Read directions carefully and show all your work. Partial credit will be assigned based upon the correctness, completeness, and clarity of your answers. Correct answers without proper justification or those that use unapproved short-cut methods will not receive full credit. If you use a calculator to help find an answer, you must write down enough information on what you have done to make your method understandable.

Good Luck!

Formulas for Common Geometric Shapes • Circle: $A = \pi r^2$, $C = 2\pi r$ • Trapezoid: $A = \frac{1}{2}h(b_1 + b_2)$

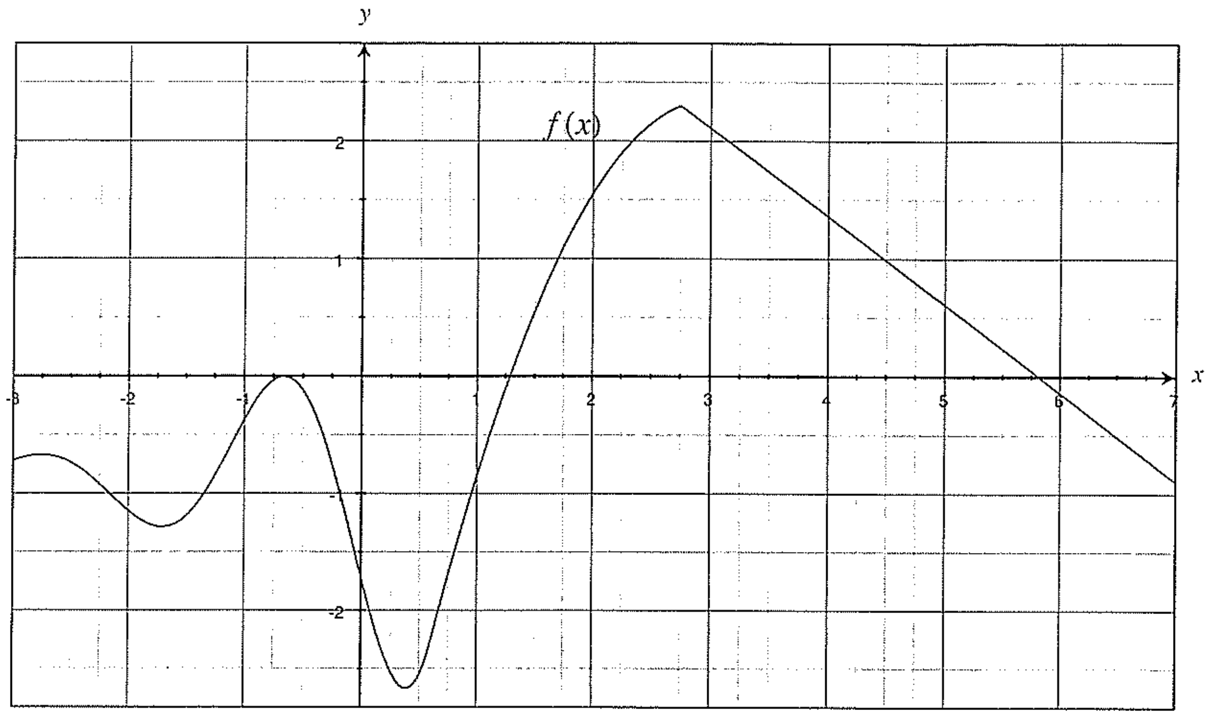
1. (6 points) The following is a graph of $f(x)$ on the interval $[-3, 7]$.



(a) Estimate $\lim_{h\to 0} \frac{f(-0.6+h) - f(-0.6)}{h}$. $\approx f'(-.6) \approx 0$

(b) Use the graph to visually estimate $f'(2)$. $\approx \frac{1.5 - .75}{2 - 1.5} = \frac{.75}{.5} = 1.5$

2. (14 points) The following is a graph of $f(x)$ on the interval $[-3, 7]$.



Let F be an antiderivative of f.

(a) Find the x-values of the critical points of F. when $F' = f = 0$ or is undefined

$x = -.7, 1.3, 5.8$

(b) Classify your critical points from (a) as local maxima, local minima, or neither.

$F' = f$ ⊖ ⊖ ⊕ ⊖

$-.7$ 1.3 5.8

F ↘ ↘ ↗ ↘

so $-.7$ neither

1.3 min

5.8 max

(c) Find the x-values of the **candidates** for inflection points of F. (You do NOT need to determine whether or not the candidates are actually inflection points.)

when $F'' = f' = 0$ or is undefined

$x = -2.75, -1.7, -.7, .35, 2.75$

(d) On what interval(s) is $f''(x)$ positive?

when $f(x)$ is concave up

$(-2.2, -1.25) \cup (0, 0.8)$

3. (4 points) Imagine it is a hot summer day, and you grab a nice cold soda from the fridge and take it outside where the temperature is 90 degrees Fahrenheit. Let $F(t)$ represent the temperature (in degrees Fahrenheit) of the soda at time t minutes after you take it out of the fridge.

(a) What units are associated with $F'(t)$?

degree F / min

(b) Is $F'(4) > 0$? Explain your answer (use common sense).

Yes, because the temperature is increasing.

4. (4 points) Suppose that $y = -7x + 8$ is the equation of the line tangent to the curve $f(x)$ at $x = 4$.

(a) Evaluate $f(4)$.

The line and the function touch when $x=4$

so $f(4) = -7(4) + 8$

$= -28 + 8 = \boxed{-20}$

(b) Evaluate $f'(4)$.

The slope of the tangent line equals $f'(4)$. And the slope of the tangent line is $\boxed{-7}$

5. (15 points) Find the derivatives for each of the following.

(a) $f(x) = x^{2/3}\arcsin(3x) - \cos 2$ [You do NOT need to simplify your answer.]

$$f'(x) = \frac{2}{3}x^{-1/3}\arcsin(3x) + x^{2/3}\frac{1}{\sqrt{1-(3x)^2}}(3)$$

(b) $A(r) = 4\ln(3^r + \cos(r^2))$ [You do NOT need to simplify your answer.]

$$A'(r) = (4)\frac{1}{3^r + \cos(r^2)}\left(\ln 3 \cdot 3^r - \sin(r^2)(2r)\right)$$

(c) $g(x) = \int_{100}^{x} 7e^{t^2}\,dt$ [You do NOT need to simplify your answer.]

$$g'(x) = 7e^{x^2}$$

6. (5 points) Find $\dfrac{dy}{dx}$ if $e^{2y} + 5x = \sqrt{x} - \dfrac{y}{x}$.

$$e^{2y} 2 \frac{dy}{dx} + 5 = \frac{1}{2} x^{-1/2} - \frac{x \frac{dy}{dx} - y}{x^2}$$

$$2e^{2y} \frac{dy}{dx} + 5 = \frac{1}{2} x^{-1/2} - \left(\frac{1}{x} \frac{dy}{dx} - \frac{y}{x^2} \right)$$

$$2e^{2y} \frac{dy}{dx} + \frac{1}{x} \frac{dy}{dx} = \frac{1}{2} x^{-1/2} + \frac{y}{x^2} - 5$$

$$\left(2e^{2y} + \frac{1}{x} \right) \frac{dy}{dx} = \frac{1}{2} x^{-1/2} + \frac{y}{x^2} - 5$$

$$\boxed{\frac{dy}{dx} = \frac{\frac{1}{2} x^{-1/2} + \frac{y}{x^2} - 5}{2e^{2y} + \frac{1}{x}}}$$

7. (5 points) Compute $\displaystyle\int_1^2 \left(\frac{x^3}{5} - 2 + \frac{3}{x^7} \right) dx.$ $= 3x^{-7}$

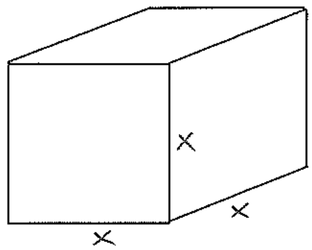
$$= \left[\frac{x^4}{4 \cdot 5} - 2x + \frac{3}{-6} x^{-6} \right]_1^2$$

$$= \left[\frac{1}{20} x^4 - 2x - \frac{1}{2} \frac{1}{x^6} \right]_1^2$$

$$= \frac{1}{20}(2^4) - 2(2) - \frac{1}{2}\left(\frac{1}{2^6}\right) - \left[\frac{1}{20} - 2 - \frac{1}{2} \right]$$

$$= \boxed{-.7578125}$$

8. (8 points) The edges of a cube increase at a constant rate of 2cm/s. How fast is the volume of the cube changing when the volume is 125 cubic cm?



know $\frac{dx}{dt} = 2$

want $\frac{dV}{dt}$

$V = x^3$

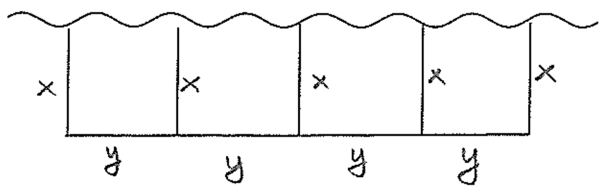
$\frac{dV}{dt} = 3x^2 \frac{dx}{dt}$

when $V = 125 = x^3$

$5 = x$

so $\frac{dV}{dt} = 3(5)^2 2 = \boxed{150 \text{ cm}^3/\text{s}}$

9. (9 points) Four equal sized pens will be built along a river by using 150 feet of fencing. What dimensions will maximize the area of the pens? Be sure to confirm, using calculus, whether you have really found the maximum.



NOTE:
There are two ways to interpret this question
① maximize the area of one pen
② maximize the area of all pens
Both approaches will ultimately give the same dimensions

Maximize area of one pen.

objective function: $A = xy$

constraint equation

$$150 = 5x + 4y$$

$$x = \frac{150 - 4y}{5} \Rightarrow x = 30 - \frac{4}{5}y$$

$$A(y) = \left(30 - \frac{4}{5}y\right)y = 30y - \frac{4}{5}y^2$$

$$A'(y) = 30 - \frac{8}{5}y$$

$$30 - \frac{8}{5}y = 0 \Rightarrow 30 = \frac{8}{5}y \Rightarrow y = \frac{75}{4}$$

$$= 18.75$$

A' is never undefined.

$A''(y) = -8/5$ which is always negative

so $y = 75/4$ is a local max of A

and $x = 30 - \frac{4}{5} \cdot \frac{75}{4} = 15$

So the dimension of one pen is $\boxed{18.75\text{ ft} \times 15\text{ ft}}$

10. (8 points) Solve the following. Only use L'Hôpital's rule when appropriate. Show your work!

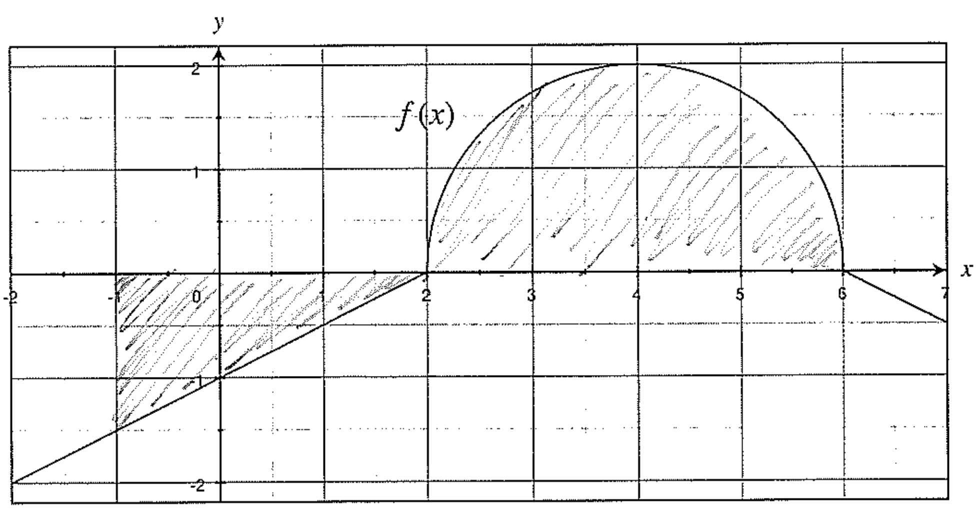
(a) $\lim_{x \to 5} \frac{x^2 - 25}{x - 5}$ $= \lim_{x \to 5} \frac{(x-5)(x+5)}{x-5} = \lim_{x \to 5} (x+5) = 10$

(b) $\lim_{n \to \infty} \frac{n^2}{4^n}$ $= \frac{\lim_{n \to \infty} n^2}{\lim_{n \to \infty} 4^n} = \frac{\infty}{\infty}$ so we can use L'Hopital's Rule

$\lim_{n \to \infty} \frac{n^2}{4^n} = \lim_{n \to \infty} \frac{2n}{4^n \ln 4} = \frac{\infty}{\infty}$ so we use L'Hopital's Rule again

$= \lim_{n \to \infty} \frac{2}{4^n (\ln 4)(\ln 4)} = \boxed{0}$

11. (4 points) The following is a graph of $f(x)$ on the interval $[-2, 7]$. $f(x)$ consists of lines and a semicircle.



Let $A(x) = \int_{-1}^{x} f(t)\, dt$. Compute $A(6)$.

$A(6) = \int_{-1}^{6} f(x)\,dx$ which is the signed area of the shaded region

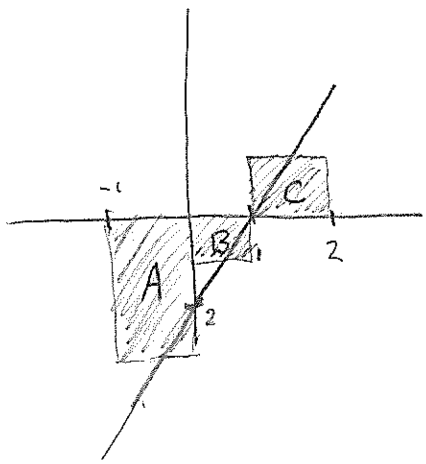
The triangular region has signed area $= \frac{1}{2}(-1.5)(3) = -2.25$

The semicircle region has signed area $= \frac{1}{2}\pi(2^2) = 2\pi \approx 6.28$

so $A(6) = -2.25 + 6.28 = \boxed{4.03}$

12. (18 points) Let $f(x) = 2x - 2$.

(a) Estimate $\int_{-1}^{2} f(x)\,dx$ using M_3, i.e., 3 rectangles and midpoint sums.



$$\Delta x = \frac{2 - -1}{3} = 1$$

$$A = \left[2\left(-\tfrac{1}{2}\right) - 2\right](1) = -3$$

$$B = \left[2\left(\tfrac{1}{2}\right) - 2\right](1) = -1$$

$$C = \left[2\left(\tfrac{3}{2}\right) - 2\right](1) = 1$$

$$\int_{-1}^{2} f(x)\,dx \approx M_3 = -3 - 1 + 1 = \boxed{-3}$$

(b) Determine $\int_{-1}^{2} f(x)\,dx$ using the FTC.

$$\int_{-1}^{2} (2x-2)\,dx = \left[x^2 - 2x\right]_{-1}^{2} = \left[2^2 - 2(2)\right] - \left[(-1)^2 - 2(-1)\right]$$

$$= 0 - [1 + 2] = \boxed{-3}$$

(8) (c) Use the limit definition of the definite integral to compute $\int_{-1}^{2} f(x)\,dx$, where $f(x) = 2x - 2$.

$$\sum_{i=1}^{n} 1 = n \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \qquad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Delta x = \frac{2--1}{n} = \frac{3}{n}$$

$$R_n = \sum_{i=1}^{n} f\left(-1+\frac{3}{n}i\right)\left(\frac{3}{n}\right) = \sum_{i=1}^{n}\left[2\left(-1+\frac{3}{n}i\right)-2\right]\frac{3}{n} = \sum_{i=1}^{n}\left[-4+\frac{6}{n}i\right]\left(\frac{3}{n}\right)$$

$$= \sum_{i=1}^{n}\left(-\frac{12}{n}+\frac{18}{n^2}i\right) = \sum_{i=1}^{n} -\frac{12}{n} + \sum_{i=1}^{n}\frac{18}{n^2}i$$

$$= -\frac{12}{n}\sum_{i=1}^{n} 1 + \frac{18}{n^2}\sum_{i=1}^{n} i = -\frac{12}{n}(n) + \frac{18}{n^2}\left(\frac{n(n+1)}{2}\right)$$

$$= -12 + \frac{9}{n}(n+1) = -12 + 9 + \frac{9}{n} = -3 + \frac{9}{n} = R_n$$

$$\int_{-1}^{2} f(x)\,dx = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(-3+\frac{9}{n}\right) = \boxed{-3}$$