Math 105: Review for Final Exam, Part II - SOLUTIONS

1. **Consider the function $f(x) = x^6 - 2x^3$ on the interval $[-2, 2]$.**

   (a) **Find the x- and y-coordinates of any and all local extrema and classify each as a local maximum or local minimum.**

$$f'(x) = 6x^5 - 6x^2$$
$$0 = 6x^2(x^3 - 1)$$
$$\Rightarrow x = 0, 1$$

	$-2 \le x < 0$	$0 < x < 1$	$1 < x \le 2$
f'	negative	negative	positive
f	↘	↘	↗

y-values: $f(0) = 0$, $f(1) = -1$

So, f has a local minimum at $(1, -1)$; $(0, 0)$ is not a local extremum.

   (b) **Find the x- and y-coordinates of any and all global extrema and classify each as a global maximum or global minimum.**

We check the y-values at the local extrema and the endpoints.

y-values: $f(-2) = 80$, $f(1) = -1$, $f(2) = 48$

So, f has a global minimum at $(1, -1)$ and a global maximum at $(-2, 80)$.

   (c) **Find the x-coordinate(s) of any and all inflection points.**

$$f''(x) = 30x^4 - 12x$$
$$0 = 6x(5x^3 - 2)$$
$$\Rightarrow x = 0, \sqrt[3]{0.4}$$

	$x < 0$	$0 < x < \sqrt[3]{0.4}$	$\sqrt[3]{0.4} < x$
f''	positive	negative	positive
f	concave up	concave down	concave up

So, the x-values of the inflection points of f are $x = 0$ and $x = \sqrt[3]{0.4}$.

2. **Your company is mass-producing a cylindrical container. The flat portion (top and bottom) costs 3 cents per square inch and the curved (lateral) portion costs 5 cents per square inch. If your budget is $9.00 per container, what dimensions will give the largest volume? [Students in the 1:10 section may omit this problem.]**

area of circle = πr^2 **lateral area of cylinder = $2\pi rh$** **volume of cylinder = $\pi r^2 h$**

Objective function: volume $= V = \pi r^2 h$

We need to get this down to a function of just one variable, so we use the

$$\textit{constraint equation}: \text{cost} = 900 = 3 \cdot 2 \cdot \pi r^2 + 5 \cdot 2\pi rh$$
$$900 = 6\pi r^2 + 10\pi rh$$
$$900 - 6\pi r^2 = 10\pi rh$$
$$\frac{900 - 6\pi r^2}{10\pi r} = h$$

Substituting this back into the objective function gives

$V = \pi r^2 h = \pi r^2 \cdot \dfrac{900 - 6\pi r^2}{10\pi r} = r \cdot \dfrac{900 - 6\pi r^2}{10} = \dfrac{1}{10}(900r - 6\pi r^3).$

Now that we have V as a function of just one variable, we find its maximum.

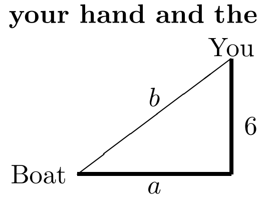
$$\begin{aligned} V'(x) &= \frac{1}{10}(900 - 18\pi r^2) \\ 0 &= \frac{1}{10}(900 - 18\pi r^2) \\ &\Rightarrow 18\pi r^2 = 900 \\ &\Rightarrow r^2 = \frac{50}{\pi} \\ &\Rightarrow r = \sqrt{\frac{50}{\pi}} \end{aligned}$$

	$0 < x < \sqrt{50/\pi}$	$\sqrt{50/\pi} < x$
f'	positive	negative
f	↗	↘

Thus, we have in fact found the global maximum at $r = \sqrt{50/\pi}$.

And $h = \dfrac{900 - 6\pi r^2}{10\pi r}$ = ...much simplifying... $= \sqrt{\dfrac{72}{\pi}} \approx 4.787$ inches.

3. **You are standing on a pier, 6 feet above the deck of a boat. Attached to the boat is a line, which you are pulling in at a rate of 3 feet per second. When there are 10 feet of line between your hand and the boat, at what rate is the boat moving across the water?**



We know $\dfrac{db}{dt}$, and we want to find $\dfrac{da}{dt}$.

So, we write an equation that relates a and b and then differentiate implicitly with respect to time t.

$$\begin{aligned} a^2 + 6^2 &= b^2 \\ 2a\frac{da}{dt} + 0 &= 2b\frac{db}{dt} \\ \frac{da}{dt} &= \frac{b}{a}\frac{db}{dt} \end{aligned}$$

At the moment in question, $b = 10$, $a = 8$ (by the Pythagorean Theorem), and $\dfrac{db}{dt} = -3$.

So, $\dfrac{da}{dt} = \dfrac{10}{8} \cdot (-3) = -3.75$ feet per second, meaning the boat is moving toward the dock at 3.75 feet per second.

4. **Use the Intermediate Value Theorem to show that $f(x) = x^3 - 2x - 1$ has a root on $[1, 2]$.**

   IVT: If f is continuous on $[a, b]$ and y is a number between $f(a)$ and $f(b)$, then there is a number c between a and b such that $f(c) = y$.

   For the function given above, $f(1) = -2$ and $f(2) = 3$. Since 0 is a number between -2 and 3, the IVT says there is a number c between 1 and 2 such that $f(c) = 0$; this c is the desired root.

5. **What (if anything) does the Extreme Value Theorem say about $f(x) = x^2$ on each of the following intervals?**

   EVT: If f is continuous on $[a, b]$, then f has both a maximum and a minimum on $[a, b]$.

(a) **$[1, 4]$**

f has a maximum and a minimum on $[1, 4]$

(b) **$(1, 4)$**

The EVT doesn't apply because $(1, 4)$ is not a closed interval since its endpoints are not included.

6. **Find the value of the constant c that the Mean Value Theorem specifies for $f(x) = x^3 + x$ on $[0, 3]$.**

MVT: If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c between a and b such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

For our function, we have $\dfrac{f(3) - f(0)}{3 - 0} = \dfrac{30 - 0}{3} = 10$.

And $f'(x) = 3x^2 + 1$, so $f'(c) = 3c^2 + 1$.

So, we solve $3c^2 + 1 = 10$, which means $c = \sqrt{3}$. (The other solution, $x = -\sqrt{3}$, is not in our interval $[0, 3]$.)

7. **Water is leaking out of a tank at a decreasing rate $r(t)$ as shown below.**

time (min)	**0**	**2**	**4**	**6**	**8**
rate (gal/min)	**15**	**11**	**8**	**4**	**3**

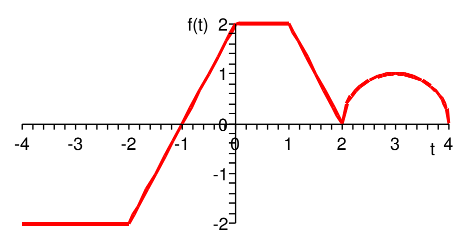
(a) **Find an overestimate and underestimate for the total amount that leaked out during these 8 minutes.**

$$\text{overestimate} = L_4 = (15 + 11 + 8 + 4)(2) = 76$$
$$\text{underestimate} = R_4 = (11 + 8 + 4 + 3)(2) = 52$$

(b) **Interpret the expression $\int_2^6 r(t)\,dt$ in terms of the situation described above.**

This integral gives the amount (in gallons) of water that leaked from the tank on the interval $[2, 6]$ minutes.

8. **Consider the graph of $f(t)$ shown. It is made of straight lines and a semicircle.**



Let $G(x) = \int_0^x f(t)\ dt$ and $H(x) = \int_{-3}^x f(t)\ dt$.

(a) **Compute $G(2)$, $G(4)$, and $H(4)$.**

$G(2)$ is the area under f between $t = 0$ and $t = 2$. This is a rectangle plus a triangle and has area $2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 = 3$.

Similarly, $G(4) = 2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1 + \frac{1}{2}\pi(1)^2 = 3 + \frac{\pi}{2}$.

$H(4)$ is the area under f between $t = -3$ and $t = 4$. Remember that area below the t-axis counts as negative.

$$\begin{aligned} H(4) &= -(2 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 1) + \frac{1}{2} \cdot 2 \cdot 1 + [\text{area under } f \text{ from 0 to 4, found above as } G(4)] \\ &= -2 + \left[3 + \frac{\pi}{2}\right] \\ &= 1 + \frac{\pi}{2} \end{aligned}$$

(b) **Where is G increasing? Where is G decreasing?**

For parts (b), (c), and (d), recall that we learned in class that $G' = f$.

G is increasing where f is positive: $(-1, 4]$. Note that G has a horizontal slope at $x = 2$ but since f is positive on each side of $t = 2$, we say G is increasing at $x = 2$.

G is decreasing where f is negative: $[-4, -1)$.

(c) **Where is G concave up? Where is G concave down?**

G is concave up where f is increasing: $(-2, 0) \cup (2, 3)$.

G is concave down where f is decreasing: $(1, 2) \cup (3, 4]$.

(d) **At what x-value(s) does G have a local maximum? At what x-value(s) does G have a local minimum?**

G has a local maximum where f changes from positive to negative: never.

G has a local minimum where f changes from negative to positive: $x = -1$.

(e) **Find a formula that relates G and H.**

From their definitions, $H(x) = \int_{-3}^{0} f(t)\, dt + G(x) = -2 + G(x)$.

(f) **How would your answers to (b), (c), and (d) change if the questions were about H instead of G?**

They would not change at all because $H'(x) = G'(x)$.

9. (a) **Use sigma notation to express L_{10} and M_{10} as approximations to $\int_{20}^{60} \ln x\, dx$.**

We're subdividing the interval into 10 pieces, so each piece has width $\Delta x = \frac{60 - 20}{10} = 4$.

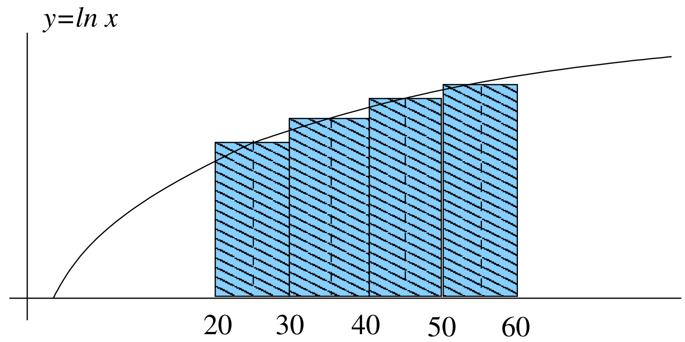
$$\begin{aligned} L_{10} &= [f(20) + f(24) + f(28) + ... + f(52) + f(56)]\Delta x \\ &= [\ln(20) + \ln(24) + \ln(28) + ... + \ln(52) + \ln(56)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(20 + 4k) \cdot 4 \end{aligned}$$

$$\begin{aligned} M_{10} &= [f(22) + f(26) + f(30) + ... + f(54) + f(58)]\Delta x \\ &= [\ln(22) + \ln(26) + \ln(30) + ... + \ln(54) + \ln(58)] \cdot 4 \\ &= \sum_{k=0}^{9} \ln(22 + 4k) \cdot 4 \end{aligned}$$

(b) **Draw a sketch that represents the sum M_4.**

Now we're subdividing the interval into 4 pieces, so each piece has width $\Delta x = \frac{60 - 20}{4} = 10$.

Note that the height of each rectangle is determined by the y-value of the curve at the *middle* x-value of the rectangle (that is, at $x = 25$, 35, 45, 55).



10. **Find the following.**

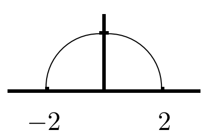
(a) **all antiderivatives of** $\mathbf{1+2x+x^3+4\sqrt{x}+\frac{1}{x^5}}$

Any such antiderivative will take the form $x+x^2+\frac{x^4}{4}+4\frac{x^{3/2}}{3/2}+\frac{x^{-4}}{-4}+C$.

Note that we have used the facts that $\sqrt{x}=x^{1/2}$ and $1/x^5=x^{-5}$.

(b) $\displaystyle\int_1^7 \frac{3}{x}\,dx = 3\ln|x|\Big|_1^7 = 3\ln 7 - 3\ln 1 = 3\ln 7$

(c) $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx = \frac{1}{2}\pi(2)^2 = 2\pi$ This integral represents the area of a semicircle of radius 2.



(d) $\displaystyle\frac{d}{dx}\int_1^x \sin\sqrt{t}\,dt = \sin\sqrt{x}$ The derivative of the area function is the original function.