Math 106: Review for Final Exam, Part I - SOLUTIONS

1. **Find the following.** [See Review for Exam II for integration tips and strategies.]

(a) Let $u = x^3$, so $du = 3x^2dx$ and $du/3 = x^2dx$.

$$\begin{aligned}\int \mathbf{12x^2\cos(x^3)\,dx} &= 12\int \cos(x^3)x^2\,dx \\ &= 12\int \cos(u)\frac{du}{3} \\ &= 4\sin(u)+C \\ &= 4\sin(x^3)+C\end{aligned}$$

(b) We'll use integration by parts: $u = x \Rightarrow du = dx$ and $dv = e^{-3x} \Rightarrow v = \dfrac{e^{-3x}}{-3}$.

$$\begin{aligned}\int_0^\infty \mathbf{xe^{-3x}\,dx} &= \lim_{t\to\infty}\int_0^t xe^{-3x}\,dx \\ &= \lim_{t\to\infty}\left[x\frac{e^{-3x}}{-3}\Big|_0^t - \int_0^t \frac{e^{-3x}}{-3}\,dx\right] \\ &= \lim_{t\to\infty}\left[x\frac{e^{-3x}}{-3} - \frac{e^{-3x}}{9}\right]_0^t \\ &= \lim_{t\to\infty}\left[\frac{-x}{3e^{3x}} - \frac{1}{9e^{3x}}\right]_0^t \\ &= \lim_{t\to\infty}\left[\frac{-t}{3e^{3t}} - \frac{1}{9e^{3t}}\right] - \left[\frac{0}{3e^0} - \frac{1}{9e^0}\right] \\ &= (0-0)-(0-1/9) \\ &= 1/9\end{aligned}$$

So, the integral converges (to this value).

(c) This integral is improper at $x = 4$ because the integrand has a vertical asymptote there, so we split into two integrals.

$$\begin{aligned}\int_0^6 \mathbf{\frac{dx}{(x-4)^2}} &= \int_0^4 \frac{dx}{(x-4)^2} + \int_4^6 \frac{dx}{(x-4)^2} \\ &= \lim_{a\to 4^-}\int_0^a \frac{dx}{(x-4)^2} + \lim_{b\to 4^+}\int_b^6 \frac{dx}{(x-4)^2} \\ &= \lim_{a\to 4^-}\frac{-1}{(x-4)}\Big|_0^a + \lim_{b\to 4^+}\frac{-1}{(x-4)}\Big|_b^6 \\ &= \lim_{a\to 4^-}\left[\frac{-1}{(a-4)} - \frac{-1}{(0-4)}\right] + \lim_{b\to 4^+}\left[\frac{-1}{(6-4)} - \frac{-1}{(b-4)}\right]\end{aligned}$$

$$\int u^{-2}\,du = -u^{-1}+C$$

Since $\lim_{a\to 4^-}\dfrac{-1}{(a-4)} = \infty$ and $\lim_{b\to 4^+} -\dfrac{-1}{(b-4)} = \infty$, this integral diverges (to ∞).

(d) Partial Fractions:

Write $\dfrac{3x^2+2x-5}{(x^2+1)(x-4)} = \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x-4}$. Now multiply both sides by $(x^2+1)(x-4)$ to get

$$3x^2+2x-5 = (Ax+B)(x-4)+C(x^2+1).$$

Let $x = 4$. Then $51 = C(17)$, so $C = 3$.
Let $x = 0$. Then $-5 = B(-4)+3(1)$, so $B = 2$.

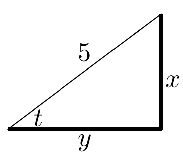
Let $x = 1$. Then $0 = (A(1) + 2)(-3) + 3(2)$, so $A = 0$.

$$\begin{aligned} \int \frac{\mathbf{3x^2 + 2x - 5}}{\mathbf{(x^2+1)(x-4)}}\, \mathbf{dx} &= \int \left[\frac{2}{x^2+1} + \frac{3}{x-4}\right]\, dx \\ &= 2\arctan x + 3\ln |\, x - 4\,| + D \end{aligned}$$

(e) Let $u = \sec x$, so $du = \sec x \tan x\ dx$.
New limits: $x = 0 \Rightarrow u = \sec 0 = 1/\cos 0 = 1$ and $x = \pi/3 \Rightarrow u = \sec(\pi/3) = 1/\cos(\pi/3) = 2$.

$$\begin{aligned} \int_0^{\pi/3} \mathbf{\tan^3 x \sec^5 x\, dx} &= \int_0^{\pi/3} \tan^2 x \sec^4 x \sec x \tan x\, dx && \text{Break off a } \sec x \tan x. \\ &= \int_0^{\pi/3} (\sec^2 x - 1)\sec^4 x \sec x \tan x\, dx && \text{Use } \tan^2 x = \sec^2 x - 1. \\ &= \int_1^2 (u^2 - 1)u^4\, du && \text{Change the limits. See above.} \\ &= \int_1^2 (u^6 - u^4)\, du \\ &= \left[\frac{u^7}{7} - \frac{u^5}{5}\right]_1^2 \\ &= \left[\frac{2^7}{7} - \frac{2^5}{5}\right] - \left[\frac{1^7}{7} - \frac{1^5}{5}\right] \\ &= \frac{418}{35} && \text{This is about 11.943.} \end{aligned}$$

(f) Let $x = 5\sin t$, so $dx = 5\cos t\ dt$.



$$x^2 + y^2 = 5^2 \Rightarrow y = \sqrt{25 - x^2}$$
$$\sin t = \frac{\text{opp}}{\text{hyp}} = \frac{x}{5} \Rightarrow t = \arcsin(x/5)$$
$$\cos t = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{25 - x^2}}{5} \Rightarrow 5\cos t = \sqrt{25 - x^2}$$

$$\begin{aligned} \int \mathbf{\sqrt{25 - x^2}\, dx} &= \int 5\cos t \cdot 5\cos t\ dt && \text{Use } dx \text{ and } \cos t \text{ from above.} \\ &= \int 25\cos^2 t dt \\ &= 25\int \left[\frac{1}{2} + \frac{\cos(2t)}{2}\right] dt && \text{Use } \cos^2 t = \frac{1}{2} + \frac{\cos(2t)}{2} \text{ or table \#42.} \\ &= 25\left[\frac{t}{2} + \frac{\sin(2t)}{4}\right] + C && \text{Let } u = 2t \text{ to integrate } \cos(2t). \\ &= 25\left[\frac{\arcsin(x/5)}{2} + \frac{2\sin t\cos t}{4}\right] + C && \text{Use } \sin(2t) = \sin t \cos t \text{ and } x \text{ from above.} \\ &= 25\left[\frac{\arcsin(x/5)}{2} + \frac{2 \cdot \frac{x}{5} \cdot \frac{\sqrt{25-x^2}}{5}}{4}\right] + C && \text{Use } \sin t \text{ and } \cos t \text{ from above.} \\ &= 25\left[\frac{\arcsin(x/5)}{2} + \frac{x\sqrt{25 - x^2}}{50}\right] + C \end{aligned}$$

2. **Find the best possible left, right, midpoint, trapezoidal, and Simpson's approximations to $\int_{-2}^{0} f(x)\,dx$ given the data in the table below.**

x	-2	-1.5	-1	-0.5	0
$f(x)$	2	3	6	10	11

$L_4 = (2+3+6+10)(0.5) = 10.5$ $R_4 = (3+6+10+11)(0.5) = 15$ $T_4 = 0.5(L_4 + R_4) = 12.75$

We cannot compute M_4, which would require the values of f at $x = -1.75, -1.25, -0.75$, and -0.25. Instead, we find $M_2 : M_2 = (3+10)(1) = 13$.

Finally, $S_4 = \dfrac{2M_2 + T_2}{3}$, so we need to compute $T_2 = \dfrac{L_2 + R_2}{2} = \dfrac{(2+6)(1) + (6+11)(1)}{2} = 12.5$.

Thus, $S_4 = \dfrac{2M_2 + T_2}{3} = \dfrac{2(13) + 12.5}{3} = \dfrac{77}{6}$.

3. **If you use numerical integration to estimate $\int_a^b \ln x\,dx$ (where a and b are positive), how would the following be ordered from least to greatest? L_{100}, R_{100}, M_{100}, T_{100}, $\int_a^b \ln x\,dx$.**

The integrand is increasing and concave down, so we have $L_{100} < T_{100} < \int_a^b \ln x\,dx < M_{100} < R_{100}$.

4. **Find bounds for each of the following errors if $I = \int_0^2 e^{-5x}\,dx$.**

(a) $|I - R_{100}| \leq \dfrac{K_1(b-a)^2}{2n} = \dfrac{5(2-0)^2}{2(100)} = \dfrac{1}{10}$

$K_1 =$ max of $|f'(x)|$ on $[0,2] =$ max of $5e^{-5x}$ on $[0,2] = 5$ (occurs at $x = 0$)

(b) $|I - T_{100}| \leq \dfrac{K_2(b-a)^3}{12n^2} = \dfrac{25(2-0)^3}{12(100)^2} = \dfrac{1}{600}$

$K_2 =$ max of $|f''(x)|$ on $[0,2] =$ max of $25e^{-5x}$ on $[0,2] = 25$ (occurs at $x = 0$)

(c) $|I - M_{100}| \leq \dfrac{K_2(b-a)^3}{24n^2} = \dfrac{25(2-0)^3}{24(100)^2} = \dfrac{1}{1200}$

$K_2 =$ same as in previous part

5. **If $I = \int_0^2 e^{-5x}\,dx$, how many subdivisions are required to obtain a midpoint sum approximation with error of at most 1/1,000,000?**

From part (c) above, we know that $|I - M_n| \leq \dfrac{K_2(b-a)^3}{24n^2} = \dfrac{25(2-0)^3}{24n^2} = \dfrac{25}{3n^2}$.

Thus, we want $\dfrac{25}{3n^2} \leq \dfrac{1}{1,000,000}$, which is equivalent to $\dfrac{25,000,000}{3} \leq \dfrac{n^2}{1}$.

Taking the square root of each side results in $\sqrt{25,000,000/3} \leq n$.

Since $\sqrt{25,000,000/3} = 2886.751...$, we must use at least 2887 subdivisions.

6. **Write an integral equal to the area between $y = 2x + 3$ and $y = x^2 + 7x - 3$.**

First, find where the curves intersect.

$$\begin{aligned} x^2 + 7x - 3 &= 2x + 3 \\ x^2 + 5x - 6 &= 0 \\ (x+6)(x-1) &= 0 \\ &\Rightarrow x = -6, x = 1 \end{aligned}$$

Between $x = -6$ and $x = 1$, $y = 2x + 3$ is above $y = x^2 + 7x - 3$. (Plug in $x = 0$ to check.) So, the area between them is $\int_{-6}^{1} [(2x+3) - (x^2 + 7x - 3)]\, dx$. [This equals 343/6.]

7. **Compute the arc length of $y = \sqrt{1 - x^2}$ from $x = 0$ to $x = 1/2$.**

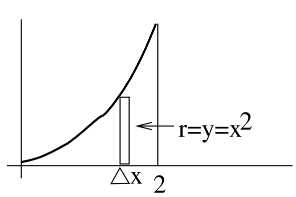
First, we find $f'(x) = \frac{1}{2}(1 - x^2)^{-1/2}(-2x) = \frac{-x}{\sqrt{1 - x^2}}$.

Thus, $[f'(x)]^2 = \frac{x^2}{1 - x^2}$.

$$\begin{aligned} \int_a^b \sqrt{1 + [f'(x)]^2}\, dx &= \int_0^{1/2} \sqrt{1 + \frac{x^2}{1 - x^2}}\, dx && \text{This is the definition of arc length.} \\ &= \int_0^{1/2} \sqrt{\frac{1 - x^2}{1 - x^2} + \frac{x^2}{1 - x^2}}\, dx && \text{Get a common denominator.} \\ &= \int_0^{1/2} \sqrt{\frac{1}{1 - x^2}}\, dx \\ &= \int_0^{1/2} \frac{\sqrt{1}}{\sqrt{1 - x^2}}\, dx \\ &= \arcsin x \Big|_0^{1/2} \\ &= \arcsin(1/2) - \arcsin(0) \\ &= \pi/6 - 0 \\ &= \pi/6 \end{aligned}$$

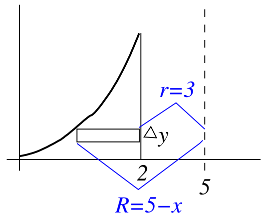
8. **Consider the region bounded by $y = 0$, $x = 2$, and $y = x^2$. Write an integral equal to the volume of the object created when the region is revolved about**

(a) **the x-axis** — Slice vertically into disks.



$$\begin{aligned} \text{volume of slice } &\approx \pi r^2 \Delta x \\ &= \pi y^2 \Delta x \\ &= \pi (x^2)^2 \Delta x \\ &= \pi x^4 \Delta x \\ \text{total volume } &= \pi \int_0^2 x^4\, dx \end{aligned}$$

(b) **the line $x = 5$** — Slice horizontally into washers.



$$\begin{aligned}
\text{volume of slice} &\approx \pi R^2 \Delta y - \pi r^2 \Delta y \\
&= \pi(5-x)^2\Delta y - \pi(3)^2 \Delta y \\
&= \pi[(5-\sqrt{y})^2 - 3^2]\Delta y \\
\text{total volume} &= \pi \int_0^4 [(5-\sqrt{y})^2 - 3^2]\, dy
\end{aligned}$$

9. **The probability density function (pdf) of the weights of newborn toads in a certain pond is given by $f(x) = \dfrac{k}{(x+1)^4}$, where x is the weight (in ounces). Note that the domain is $x \geq 0$ since no toad can have a negative weight.**

   (a) **What must be the value of k?**

   We know that the total area under any pdf must be 1 (because it must account for 100% of events.)

$$\begin{aligned}
\int_0^\infty \frac{k}{(x+1)^4}\,dx &= \lim_{t\to\infty} \int_0^t \frac{k}{(x+1)^4}\,dx \\
&= \lim_{t\to\infty} \left.\frac{k(x+1)^{-3}}{-3}\right|_0^t \\
&= \lim_{t\to\infty} \left.\frac{k}{-3(x+1)^3}\right|_0^t \\
&= \lim_{t\to\infty} \frac{k}{-3(t+1)^3} - \frac{k}{-3(0+1)^3} \\
&= 0 - \frac{k}{-3} \\
&= \frac{k}{3}
\end{aligned}$$

   So, we have $k/3 = 1$ or $k = 3$.

   (b) **What fraction of the newborn toads weigh more than one ounce?**

$$\begin{aligned}
\int_1^\infty \frac{3}{(x+1)^4\,dx} &= \lim_{t\to\infty} \int_1^t \frac{3}{(x+1)^4\,dx} \\
&= \lim_{t\to\infty} \left.\frac{3}{-3(x+1)^3}\right|_1^t \\
&= \lim_{t\to\infty} \frac{1}{-1(t+1)^3} - \frac{1}{-1(1+1)^3} \\
&= 0 - \frac{1}{-8} \\
&= \frac{1}{8}
\end{aligned}$$

   Note that we could instead have computed $1 - \displaystyle\int_0^1 \frac{3}{(x+1)^4}\,dx$ and gotten the same answer.

10. **Find the solution to $\frac{dy}{dx} = \frac{\cos x}{y^2}$ that passes through $(0, 2)$. [Students in the 9:30 section should omit this problem.]**

We use separation of variables.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos x}{y^2} \\ y^2 \ dy &= \cos x \ dx \\ \int y^2 \ dy &= \int \cos x \ dx \\ y^3/3 &= \sin x + C \\ y^3 &= 3\sin x + D \\ y &= \sqrt[3]{3\sin x + D} \end{aligned}$$

When $x = 0$, we have $y = 2$, so $2 = \sqrt[3]{3\sin 0 + D}$, or $2 = \sqrt[3]{D}$. Thus, $D = 8$.
Therefore, the solution is $y = \sqrt[3]{3\sin x + 8}$.