Math 105: Review for Final Exam, Part I - SOLUTIONS

1. **Consider the function $f(x) = \dfrac{3}{5-2x}$.**

   (a) **Is this function continuous on the interval $(-\infty, \infty)$? Explain.**
   No. f is discontinuous at $x = 2.5$, where f is undefined (and has a vertical asymptote).

   (b) **Compute the average rate of change of f on $[2, 2.01]$.**

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[\frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)}\right] \cdot \frac{1}{.01} \approx 6.122$$

   (c) **Using the limit definition of the derivative, compute $f'(x)$.**

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists}\\
&= \lim_{h\to 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h}\\
&= \lim_{h\to 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator}\\
&= \lim_{h\to 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5-2(x+h)](5-2x)h}\\
&= \lim_{h\to 0} \frac{6h}{[5-2(x+h)](5-2x)h}\\
&= \lim_{h\to 0} \frac{6}{[5-2(x+h)](5-2x)}\\
&= \frac{6}{(5-2x)^2}
\end{aligned}$$

   (d) **Find the equation of the tangent line to f at $x = 2$.**
   We want $y = mx + b$. $m = f'(2) = \dfrac{6}{(5-2(2))^2} = 6$, so $y = 6x + b$.
   [Note that this slope agrees well with our answer from (b) above.]
   When $x = 2$, $y = f(2) = \dfrac{3}{5-2(2)} = 3$.
   Thus, $3 = 6 \cdot 2 + b$, so $b = -9$ and we have $y = 6x - 9$.

2. **Given that $f(0) = 2$, $g(0) = 3$, $f'(0) = 5$, $g'(0) = 7$, and $f'(3) = \pi$ compute the following.**

   (a) **$h'(0)$ if $h(x) = f(x)g(x)$**
   $h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$

   (b) **$j'(0)$ if $j(x) = \dfrac{f(x)}{g(x)}$**

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)^2]} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

   (c) **$k'(0)$ if $k(x) = f(g(x))$**
   $k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$

3. **Compute dy/dx for each of the following.**

(a) $\boldsymbol{y = x^5 + 5^x + e^5 + \frac{x}{5} + \frac{5}{x} + \frac{5}{\sqrt[5]{x}} + \ln(5x) + \arctan(5x) + \ln(5) + \sin 5}$

$$\frac{dy}{dx} = 5x^4 + (\ln 5)5^x + 0 + \frac{1}{5} - 5x^{-2} + 5 \cdot \frac{-1}{5}x^{-6/5} + \frac{1}{5x} \cdot 5 + \frac{1}{1+(5x)^2} \cdot 5 + 0 + 0$$
$$= 5x^4 + (\ln 5)5^x + \frac{1}{5} - \frac{5}{x^2} - \frac{1}{x^{6/5}} + \frac{1}{x} + \frac{5}{1+25x^2}$$

(b) $\boldsymbol{y = \sqrt[3]{x}\cos(7x^3)}$

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3}\cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3}\sin(7x^3)$$

(c) $\boldsymbol{y = \frac{e^x + e^\pi}{\tan 4 - 7x}}$

$$\frac{dy}{dx} = \frac{e^x(\tan 4 - 7x) - (-7)(e^x + e^\pi)}{(\tan 4 - 7x)^2}$$

(d) $\boldsymbol{y = \tan\left(e^{x^2 \arcsin(5x)}\right)}$

$$\frac{dy}{dx} = \sec^2\left(e^{x^2 \arcsin(5x)}\right) \cdot e^{x^2 \arcsin(5x)} \cdot \left[x^2 \frac{1}{\sqrt{1-25x^2}} \cdot 5 + 2x \arcsin(5x)\right]$$

(e) $\boldsymbol{y^3 + yx^2 + x^2 = 3y^2}$ Here we use implicit differentiation.

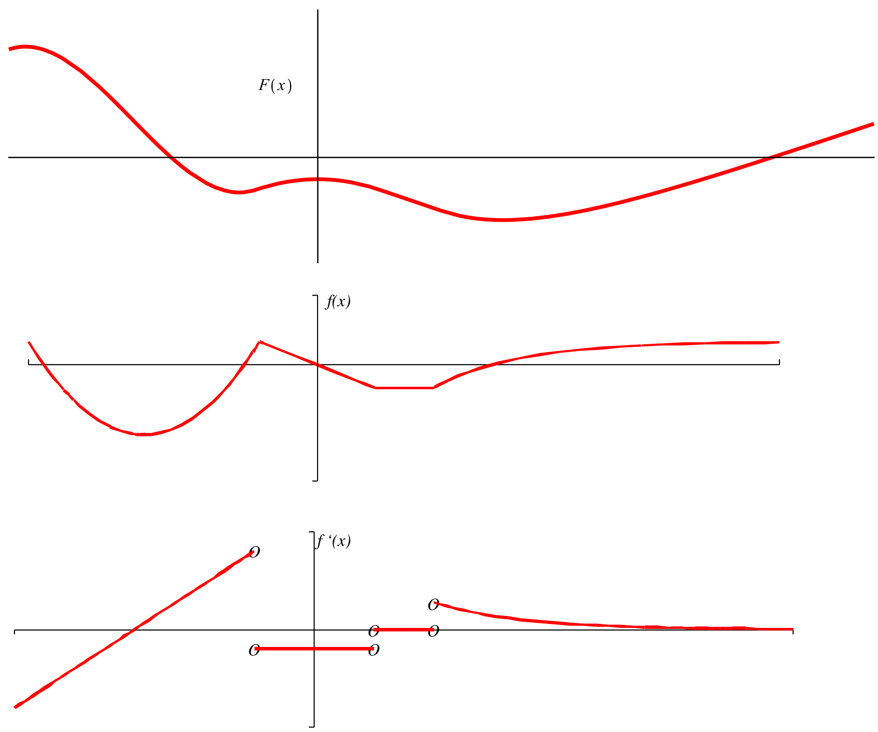
$$\begin{aligned}
3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y\frac{dy}{dx} \\
3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y\frac{dy}{dx} &= -2xy - 2x \\
\frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\
\frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y}
\end{aligned}$$

(f) $\boldsymbol{y = (x^2+1)^{\sin x}}$ **[Students in the 1:10 section may consider this as a bonus problem.]**

Since we have x in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned}
\ln y &= \ln(x^2+1)^{\sin x} \\
\ln y &= \sin x \cdot \ln(x^2+1) \\
\frac{1}{y} \cdot \frac{dy}{dx} &= \cos x \cdot \ln(x^2+1) + \sin x \cdot \frac{1}{x^2+1} \cdot 2x \\
\frac{dy}{dx} &= \left[\cos x \cdot \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right] \cdot y \\
\frac{dy}{dx} &= \left[\cos x \cdot \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right] \cdot (x^2+1)^{\sin x}
\end{aligned}$$

4. **Given the graph of $\boldsymbol{f}$, sketch a graph of $\boldsymbol{f'}$ and a graph of $\boldsymbol{F}$, an antiderivative of $\boldsymbol{f}$ such that $\boldsymbol{F(0) = -1}$.**



Note: The concave up portion on the left side of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f, your graph of f' may be concave up/down there.

5. **Shown below is a graph of $\boldsymbol{f'}$ on its entire domain. The graph is NOT $\boldsymbol{f}$.**

At which $\boldsymbol{x}$-value(s)

(a) **does $\boldsymbol{f}$ have a stationary point?** c, f, h

(b) **does $\boldsymbol{f}$ have a local max?** c, h

(c) **does $\boldsymbol{f}$ have a local min?** f

(d) **does $\boldsymbol{f'}$ have a stationary point?** b, d, g, i

(e) **does $\boldsymbol{f'}$ have a local max?** b, g

(f) **does $\boldsymbol{f'}$ have a local min?** d, i

(g) **is $\boldsymbol{f}$ greatest?** c

(h) **is $\boldsymbol{f}$ least?** j

(i) **is $\boldsymbol{f'}$ greatest?** b

(j) **is $\boldsymbol{f'}$ least?** d

(k) **is $\boldsymbol{f''}$ greatest?** e

(l) **is $\boldsymbol{f''}$ least?** c

On what interval(s) is

(a) **$\boldsymbol{f}$ increasing?** $[a, c) \cup (f, h)$

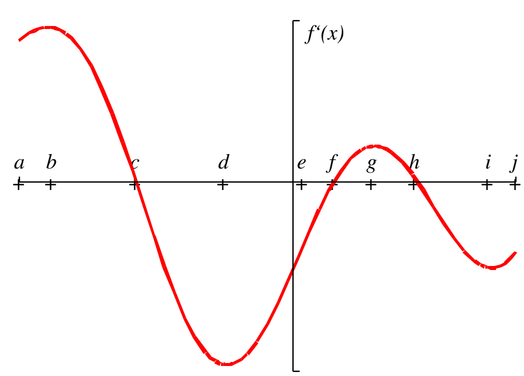
(b) **$\boldsymbol{f}$ decreasing?** $(c, f) \cup (h, j]$

(c) **$\boldsymbol{f'}$ increasing?** $[a, b) \cup (d, g) \cup (i, j]$

(d) **$\boldsymbol{f'}$ decreasing?** $(b, d) \cup (g, i)$

(e) **$\boldsymbol{f}$ concave up?** $[a, b) \cup (d, g) \cup (i, j]$

(f) **$\boldsymbol{f}$ concave down?** $(b, d) \cup (g, i)$



6. **Is $y = 7e^{3x}$ a solution to the differential equation $y'' + 2y' - 15y = 0$? Explain.**

A given function y will be a solution to the differential equation if, when we substitute in y'', y', and y, the equation is satisfied (that is, both sides of it are equal).

Since $y = 7e^{3x}$, we know that $y' = 21e^{3x}$ and $y'' = 63e^{3x}$ from the Chain Rule.

Now we check to see whether our y satisfies the differential equation.

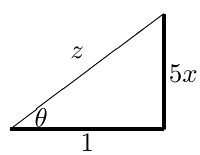
$$\begin{aligned} y'' + 2y' - 15y &\stackrel{?}{=} 0 \\ 63e^{3x} + 2 \cdot 21e^{3x} - 15 \cdot 7e^{3x} &\stackrel{?}{=} 0 \\ 63e^{3x} + 42e^{3x} - 105e^{3x} &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

So, we see that $y = 7e^{3x}$ is in fact a solution to this differential equation.

7. **Rewrite $\sin(\arctan(5x))$ as an algebraic expression. [Students in the 8:00 section may omit this problem.]**

Let $\theta = \arctan(5x)$. That is, θ is the angle whose tangent is $5x$.

We draw a triangle for which $\dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{5x}{1} = 5x$.



$$1^2 + (5x)^2 = z^2 \Rightarrow z = \sqrt{1 + 25x^2}$$

$$\sin(\arctan(5x)) = \sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{5x}{\sqrt{1 + 25x^2}}$$

8. **Evaluate the following limits.**

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\stackrel{\text{“}0/0\text{”}}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or = "0/0" or "has the form '$\frac{0}{0}$' and so, by L'Hopital's Rule, is equal to" or something else. The symbol ♡ will serve the same purpose for the indeterminate form ∞/∞.

(a) $\displaystyle \lim_{x\to\infty} \frac{x^2}{\ln x} \heartsuit \lim_{x\to\infty} \frac{2x}{1/x} = \lim_{x\to\infty} 2x^2 = \infty$

(b) $\displaystyle \lim_{x\to 0} \frac{\sin(12x) - 12x}{x^3} \bigstar \lim_{x\to 0} \frac{12\cos(12x) - 12}{3x^2} \bigstar \lim_{x\to 0} \frac{-144\sin(12x)}{6x} \bigstar \lim_{x\to 0} \frac{-1728\cos(12x)}{6} = -288$

(c) $\displaystyle \lim_{x\to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$

(d) $\displaystyle \lim_{x\to 2} \frac{x^3 - 8}{x - 2} \bigstar \lim_{x\to 2} \frac{3x^2}{1} = 12$