

Math 105: Review for Final Exam, Part I - SOLUTIONS

1. Consider the function  $f(x) = \frac{3}{5-2x}$ .

(a) Is this function continuous on the interval  $(-\infty, \infty)$ ? Explain.

No.  $f$  is discontinuous at  $x = 2.5$ , where  $f$  is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of  $f$  on  $[2, 2.01]$ .

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[ \frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122$$

(c) Using the limit definition of the derivative, compute  $f'(x)$ .

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\ &= \lim_{h \rightarrow 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6h}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6}{[5 - 2(x+h)](5-2x)} \\ &= \frac{6}{(5-2x)^2} \end{aligned}$$

(d) Find the equation of the tangent line to  $f$  at  $x = 2$ .

We want  $y = mx + b$ .  $m = f'(2) = \frac{6}{(5-2(2))^2} = 6$ , so  $y = 6x + b$ .

[Note that this slope agrees well with our answer from (b) above.]

When  $x = 2$ ,  $y = f(2) = \frac{3}{5-2(2)} = 3$ .

Thus,  $3 = 6 \cdot 2 + b$ , so  $b = -9$  and we have  $y = 6x - 9$ .

2. Given that  $f(0) = 2$ ,  $g(0) = 3$ ,  $f'(0) = 5$ ,  $g'(0) = 7$ , and  $f'(3) = \pi$  compute the following.

(a)  $h'(0)$  if  $h(z) = f(z)g(z)$

$$h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$$

(b)  $j'(0)$  if  $j(z) = \frac{f(z)}{g(z)}$

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

(c)  $k'(0)$  if  $k(z) = f(g(z))$

$$k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$$

3. (a) Find  $\frac{dy}{dt}$  if  $y = t^5 + 5^t + e^5 + \frac{t}{5} + \frac{5}{t} + \frac{5}{\sqrt[5]{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin 5$ .

$$\begin{aligned}\frac{dy}{dt} &= 5t^4 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{-1}{5}t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1+(5t)^2} \cdot 5 + 0 + 0 \\ &= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{6/5}} + \frac{1}{t} + \frac{5}{1+25t^2}\end{aligned}$$

- (b) Find  $\frac{dy}{dx}$  if  $y = \sqrt[3]{x} \cos(7x^3)$ .

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)$$

- (c) Find  $\frac{dy}{dz}$  if  $y = \frac{e^z + e^\pi}{\tan 4 - 7z}$ .

$$\frac{dy}{dz} = \frac{e^z(\tan 4 - 7z) - (-7)(e^z + e^\pi)}{(\tan 4 - 7z)^2}$$

- (d) Find  $\frac{dy}{dr}$  if  $y = \tan(e^{r^2} \arcsin(5r))$ .

$$\frac{dy}{dr} = \sec^2(e^{r^2} \arcsin(5r)) \cdot e^{r^2} \arcsin(5r) \cdot \left[ r^2 \frac{1}{\sqrt{1-25r^2}} \cdot 5 + 2r \arcsin(5r) \right]$$

- (e) Find  $\frac{dy}{dx}$  if  $y^3 + yx^2 + x^2 = 3y^2$ .

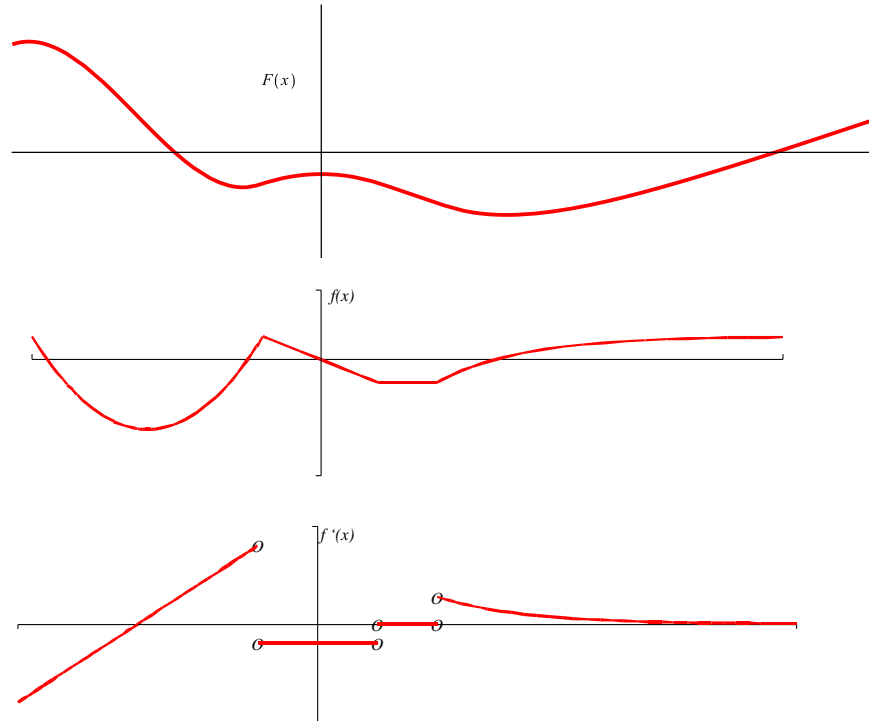
Here we use implicit differentiation.

$$\begin{aligned}3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y \frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y}\end{aligned}$$

- (f) Find  $\frac{dy}{dt}$  if  $y = (1 + x^6)^{8x}$ . Since we have  $x$  in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned}\ln y &= 8x \ln(1 + x^6) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5 \\ \frac{dy}{dx} &= \left[ 8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot y \\ \frac{dy}{dx} &= \left[ 8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot (1 + x^6)^{8x}\end{aligned}$$

4. Given the graph of  $f$ , sketch a graph of  $f'$  and a graph of  $F$ , an antiderivative of  $f$  such that  $F(0) = -1$ .

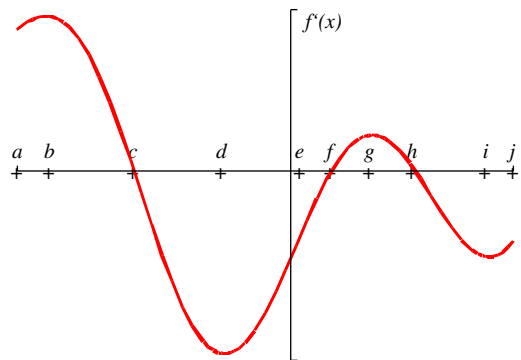


Note: The concave up portion on the left side of the graph of  $f$  is a perfect parabola, so its derivative ( $f'$ ) is linear; since you don't know the equation for  $f$ , your graph of  $f'$  may be concave up/down there.

5. Shown below is a graph of  $f'$  on its entire domain. The graph is NOT  $f$ .

At which  $x$ -value(s)

- |   |   |
|---|---|
| (a) does $f$ have a stationary point? $c, f, h$     | (b) $f$ decreasing? $(c, f) \cup (h, j]$              |
| (b) does $f$ have a local max? $c, h$               | (c) $f'$ increasing? $[a, b) \cup (d, g) \cup (i, j]$ |
| (c) does $f$ have a local min? $f$                  | (d) $f'$ decreasing? $(b, d) \cup (g, i)$             |
| (d) does $f'$ have a stationary point? $b, d, g, i$ | (e) $f$ concave up? $[a, b) \cup (d, g) \cup (i, j]$  |
| (e) does $f'$ have a local max? $b, g$              | (f) $f$ concave down? $(b, d) \cup (g, i)$            |
| (f) does $f'$ have a local min? $d, i$              |   |
| (g) is $f$ greatest? $c$                            |   |
| (h) is $f$ least? $j$                               |   |
| (i) is $f'$ greatest? $b$                           |   |
| (j) is $f'$ least? $d$                              |   |
| (k) is $f''$ greatest? $e$                          |   |
| (l) is $f''$ least? $c$                             |   |



On what interval(s) is

- (a)  $f$  increasing?  $[a, c) \cup (f, h)$

6. Solve the IVP  $y' = e^x - \sin x + 5$  given that  $y(0) = 3$ . [Students in the 8:00 section may omit this problem.]

We antidifferentiate each side to obtain  $y(x) = e^x + \cos x + 5x + C$ . To find  $C$ , we let  $x = 0$ , meaning  $3 = e^0 + \cos 0 + 5 \cdot 0 + C$ , so  $C = 1$  and our solution is  $y(x) = e^x + \cos x + 5x + 1$ .

7. Evaluate the following limits.

Throughout this solution, the symbol  $\star$  will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form  $0/0$ ; this may be  $\stackrel{0/0}{=}$  or  $\stackrel{L'H}{=}$  or  $\stackrel{H}{=}$  or  $\stackrel{0/0}{\sim}$  or "has the form  $\frac{0}{0}$ , and so, by L'Hopital's Rule, is equal to" or something else. The symbol  $\heartsuit$  will serve the same purpose for the indeterminate form  $\infty/\infty$ .

- (a)  $\lim_{x \rightarrow \infty} \frac{x^2}{\ln x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{1/x} = \lim_{x \rightarrow \infty} 2x^2 = \infty$
- (b)  $\lim_{z \rightarrow 0} \frac{\sin(5z) - 5z}{z^3} \star \lim_{z \rightarrow 0} \frac{5 \cos(5z) - 5}{3z^2} \star \lim_{z \rightarrow 0} \frac{-25 \sin(5z)}{6z} \star \lim_{z \rightarrow 0} \frac{-125 \cos(12z)}{6} = -\frac{125}{6}$
- (c)  $\lim_{x \rightarrow 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$
- (d)  $\lim_{r \rightarrow 2} \frac{r^3 - 8}{r - 2} \star \lim_{r \rightarrow 2} \frac{3r^2}{1} = 12$

8. Consider the function  $f(x) = x^6 - 2x^3$  on the interval  $[-2, 2]$ .

- (a) Find the  $x$ - and  $y$ -coordinates of any and all local extrema and classify each as a local maximum, local minimum, or neither.

$$f'(x) = 6x^5 - 6x^2 \quad \text{Since } f'(x) \text{ never fails to exist, we just need to solve } f'(x) = 0.$$

$$0 = 6x^2(x^3 - 1)$$

$$\Rightarrow x = 0, 1$$

	$-2 \leq x < 0$	$0 < x < 1$	$1 < x \leq 2$
$f'$	negative	negative	positive
$f$	↘	↘	↗

$y$ -values:  $f(0) = 0$ ,  $f(1) = -1$

So,  $f$  has a local minimum at  $(1, -1)$ ;  $(0, 0)$  is not a local extremum.

- (b) Find the  $x$ - and  $y$ -coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the  $y$ -values at the local extrema and the endpoints.

$y$ -values:  $f(-2) = 80$ ,  $f(1) = -1$ ,  $f(2) = 48$

So,  $f$  has a global minimum at  $(1, -1)$  and a global maximum at  $(-2, 80)$ .

- (c) Find the  $x$ -coordinate(s) of any and all inflection points.

$$f''(x) = 30x^4 - 12x \quad \text{Since } f''(x) \text{ never fails to exist, we just need to solve } f''(x) = 0.$$

$$0 = 6x(5x^3 - 2)$$

$$\Rightarrow x = 0, \sqrt[3]{0.4}$$

	$x < 0$	$0 < x < \sqrt[3]{0.4}$	$\sqrt[3]{0.4} < x$
$f''$	positive	negative	positive
$f$	concave up	concave down	concave up

So, the  $x$ -values of the inflection points of  $f$  are  $x = 0$  and  $x = \sqrt[3]{0.4}$ .