

Math 105: Review for Final Exam, Part I - SOLUTIONS

1. Consider the function $f(x) = \frac{3}{5-2x}$.

(a) Is this function continuous on the interval $(-\infty, \infty)$? Explain.

No. f is discontinuous at $x = 2.5$, where f is undefined (and has a vertical asymptote).

(b) Compute the average rate of change of f on $[2, 2.01]$.

$$\frac{f(2.01) - f(2)}{2.01 - 2} = \left[\frac{3}{5 - 2(2.01)} - \frac{3}{5 - 2(2)} \right] \cdot \frac{1}{.01} \approx 6.122$$

(c) Using the limit definition of the derivative, compute $f'(x)$.

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} && \text{provided this limit exists} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3}{5-2(x+h)} - \frac{3}{5-2x}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{3(5-2x)}{[5-2(x+h)](5-2x)} - \frac{3[5-2(x+h)]}{[5-2(x+h)](5-2x)}}{h} && \text{common denominator} \\ &= \lim_{h \rightarrow 0} \frac{15 - 6x - (15 - 6x - 6h)}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6h}{[5 - 2(x+h)](5-2x)h} \\ &= \lim_{h \rightarrow 0} \frac{6}{[5 - 2(x+h)](5-2x)} \\ &= \frac{6}{(5-2x)^2} \end{aligned}$$

(d) Find the equation of the tangent line to f at $x = 2$.

We want $y = mx + b$. $m = f'(2) = \frac{6}{(5-2(2))^2} = 6$, so $y = 6x + b$.

[Note that this slope agrees well with our answer from (b) above.]

When $x = 2$, $y = f(2) = \frac{3}{5-2(2)} = 3$.

Thus, $3 = 6 \cdot 2 + b$, so $b = -9$ and we have $y = 6x - 9$.

2. Given that $f(0) = 2$, $g(0) = 3$, $f'(0) = 5$, $g'(0) = 7$, and $f'(3) = \pi$ compute the following.

(a) $h'(0)$ if $h(z) = f(z)g(z)$

$$h'(0) = f'(0)g(0) + f(0)g'(0) = (5)(3) + (2)(7) = 29$$

(b) $j'(0)$ if $j(z) = \frac{f(z)}{g(z)}$

$$j'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{[g(0)]^2} = \frac{(5)(3) - (2)(7)}{3^2} = \frac{1}{9}$$

(c) $k'(0)$ if $k(z) = f(g(z))$

$$k'(0) = f'(g(0)) \cdot g'(0) = f'(3) \cdot (7) = (\pi)(7) = 7\pi$$

3. (a) Find $\frac{dy}{dt}$ if $y = t^5 + 5^t + e^5 + \frac{t}{5} + \frac{5}{t} + \frac{5}{\sqrt[5]{t}} + \ln(5t) + \arctan(5t) + \ln(5) + \sin 5$.

$$\begin{aligned}\frac{dy}{dt} &= 5t^4 + (\ln 5)5^t + 0 + \frac{1}{5} - 5t^{-2} + 5 \cdot \frac{-1}{5}t^{-6/5} + \frac{1}{5t} \cdot 5 + \frac{1}{1+(5t)^2} \cdot 5 + 0 + 0 \\ &= 5t^4 + (\ln 5)5^t + \frac{1}{5} - \frac{5}{t^2} - \frac{1}{t^{6/5}} + \frac{1}{t} + \frac{5}{1+25t^2}\end{aligned}$$

- (b) Find $\frac{dy}{dx}$ if $y = \sqrt[3]{x} \cos(7x^3)$.

$$\frac{dy}{dx} = \frac{1}{3}x^{-2/3} \cos(7x^3) + \sqrt[3]{x}(-\sin(7x^3)(21x^2)) = \frac{\cos(7x^3)}{3x^{2/3}} - 21x^{7/3} \sin(7x^3)$$

- (c) Find $\frac{dy}{dz}$ if $y = \frac{e^z + e^\pi}{\tan 4 - 7z}$.

$$\frac{dy}{dz} = \frac{e^z(\tan 4 - 7z) - (-7)(e^z + e^\pi)}{(\tan 4 - 7z)^2}$$

- (d) Find $\frac{dy}{dr}$ if $y = \tan(e^{r^2} \arcsin(5r))$.

$$\frac{dy}{dr} = \sec^2(e^{r^2} \arcsin(5r)) \cdot e^{r^2} \arcsin(5r) \cdot \left[r^2 \frac{1}{\sqrt{1-25r^2}} \cdot 5 + 2r \arcsin(5r) \right]$$

- (e) Find $\frac{dy}{dx}$ if $y^3 + yx^2 + x^2 = 3y^2$.

Here we use implicit differentiation.

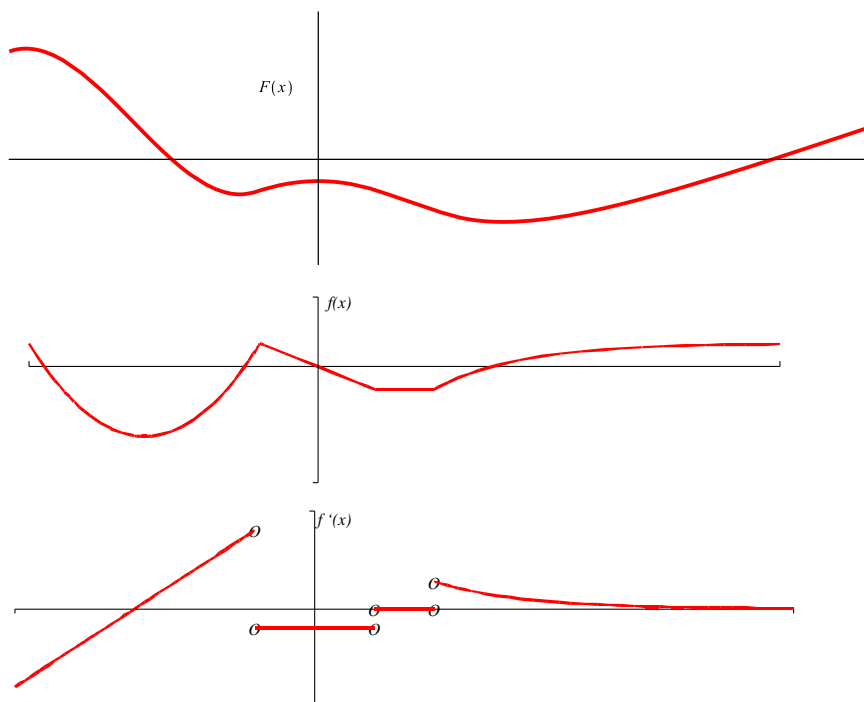
$$\begin{aligned}3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 + 2xy + 2x &= 6y \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} + \frac{dy}{dx}x^2 - 6y \frac{dy}{dx} &= -2xy - 2x \\ \frac{dy}{dx}(3y^2 + x^2 - 6y) &= -2xy - 2x \\ \frac{dy}{dx} &= \frac{-2xy - 2x}{3y^2 + x^2 - 6y}\end{aligned}$$

- (f) Find $\frac{dy}{dt}$ if $y = (1 + x^6)^{8x}$.

Since we have x in the base and the exponent, we need logarithmic differentiation.

$$\begin{aligned}\ln y &= 8x \ln(1 + x^6) \\ \frac{1}{y} \cdot \frac{dy}{dx} &= 8 \cdot \ln(1 + x^6) + 8x \cdot \frac{1}{1 + x^6} \cdot 6x^5 \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot y \\ \frac{dy}{dx} &= \left[8 \cdot \ln(1 + x^6) + \frac{48x^6}{1 + x^6} \right] \cdot (1 + x^6)^{8x}\end{aligned}$$

4. Given the graph of f , sketch a graph of f' and a graph of F , an antiderivative of f such that $F(0) = -1$



Note: The concave up portion on the left side of the graph of f is a perfect parabola, so its derivative (f') is linear; since you don't know the equation for f , your graph of f' may be concave up/down there.

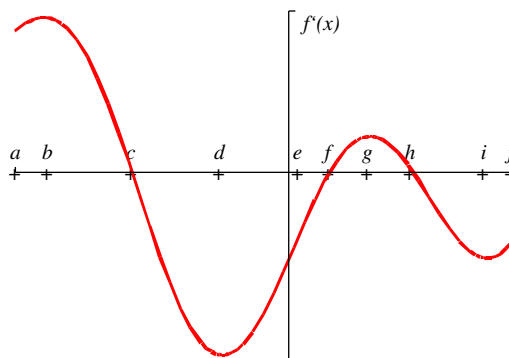
5. Shown below is a graph of f' on its entire domain. The graph is NOT f .

At which x -value(s)

- (a) does f have a stationary point? c, f, h
- (b) does f have a local max? c, h
- (c) does f have a local min? f
- (d) does f' have a stationary point? b, d, g, i
- (e) does f' have a local max? b, g
- (f) does f' have a local min? d, i
- (g) is f greatest? c
- (h) is f least? j
- (i) is f' greatest? b
- (j) is f' least? d
- (k) is f'' greatest? e
- (l) is f'' least? c

- (c) f' increasing? a to b and d to g and i to j
- (d) f' decreasing? b to d and g to i
- (e) f concave up? a to b and d to g and i to j
- (f) f concave down? b to d and g to i

* Whether to include the endpoints of these intervals will depend on your instructor's definitions.



On what interval(s)* is

- (a) f increasing? a to c and f to h
- (b) f decreasing? c to f and h to j

6. Solve the IVP $y' = e^x - \sin x + 5$ given that $y(0) = 3$.

We antidifferentiate each side to obtain $y(x) = e^x + \cos x + 5x + C$. To find C , we let $x = 0$, meaning $3 = e^0 + \cos 0 + 5 \cdot 0 + C$, so $C = 1$ and our solution is $y(x) = e^x + \cos x + 5x + 1$.

7. Evaluate the following limits.

Throughout this solution, the symbol ★ will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form 0/0; this may be $\stackrel{0/0}{=}$ or $\stackrel{L'H}{=}$ or $\stackrel{H}{=}$ or "0/0" or "has the form $\frac{0}{0}$ ", and so, by L'Hopital's Rule, is equal to" or something else. The symbol ♡ will serve the same purpose for the indeterminate form ∞/∞ .

(a) $\lim_{x \rightarrow \infty} \frac{x^2}{\ln x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{1/x} = \lim_{x \rightarrow \infty} 2x^2 = \infty$

(b) $\lim_{z \rightarrow 0} \frac{\sin(5z) - 5z}{z^3} \star \lim_{z \rightarrow 0} \frac{5 \cos(5z) - 5}{3z^2} \star \lim_{z \rightarrow 0} \frac{-25 \sin(5z)}{6z} \star \lim_{z \rightarrow 0} \frac{-125 \cos(12z)}{6} = -\frac{125}{6}$

(c) $\lim_{x \rightarrow 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0$

(d) $\lim_{x \rightarrow 0^+} x^3 \ln x$

This is of the form $0 \cdot (-\infty)$, so we rewrite it as a fraction to turn it into a L'Hopital's Rule problem.

$$\lim_{x \rightarrow 0^+} x^3 \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{\frac{1}{x^3}} \heartsuit \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{-3}{x^4}} = \lim_{x \rightarrow 0^+} \frac{1}{x} \cdot \frac{x^4}{-3} = \lim_{x \rightarrow 0^+} \frac{x^3}{-3} = \frac{0}{-3} = 0$$

8. Consider the function $f(x) = x^6 - 2x^3$ on the interval $[-2, 2]$.

- (a) Find the x - and y -coordinates of any and all local extrema and classify each as a local maximum, local minimum, or neither.

$f'(x) = 6x^5 - 6x^2$ Since $f'(x)$ never fails to exist, we just need to solve $f'(x) = 0$.

$0 = 6x^2(x^3 - 1)$

$\Rightarrow x = 0, 1$

	$-2 \leq x < 0$	$0 < x < 1$	$1 < x \leq 2$
f'	negative	negative	positive
f	↘	↘	↗

y -values: $f(0) = 0, f(1) = -1$

So, f has a local minimum at $(1, -1)$; $(0, 0)$ is not a local extremum.

- (b) Find the x - and y -coordinates of any and all global extrema and classify each as a global maximum or global minimum.

We check the y -values at the local extrema and the endpoints.

y -values: $f(-2) = 80, f(1) = -1, f(2) = 48$

So, f has a global minimum at $(1, -1)$ and a global maximum at $(-2, 80)$.

- (c) Find the x -coordinate(s) of any and all inflection points.

$f''(x) = 30x^4 - 12x$ Since $f''(x)$ never fails to exist, we just need to solve $f''(x) = 0$.

$0 = 6x(5x^3 - 2)$

$\Rightarrow x = 0, \sqrt[3]{0.4}$

	$x < 0$	$0 < x < \sqrt[3]{0.4}$	$\sqrt[3]{0.4} < x$
f''	positive	negative	positive
f	concave up	concave down	concave up

So, the x -values of the inflection points of f are $x = 0$ and $x = \sqrt[3]{0.4}$.