1. Let $\mathbf{b}_1 = \begin{bmatrix} 3 \\ 9 \\ 7 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{p}_1 = \begin{bmatrix} 6 \\ 27 \\ 26 \end{bmatrix}$, and $\mathbf{p}_2 = \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}$. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and let $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2\}$.

It's a fact that $\mathcal{B}$ and $\mathcal{P}$ are bases for the same subspace H of $\mathbb{R}^3$; you do not need to prove this.

1A: Find $[\mathbf{b}_1]_{\mathcal{P}}$. (this is the vector of weights α, β for which $\vec{b}_1 = \alpha\vec{p}_1 + \beta\vec{p}_2$, and is found using the augmented matrix

$$\left[\begin{array}{cc|c} \vec{p}_1 & \vec{p}_2 & \vec{b}_1 \end{array}\right] = \left[\begin{array}{cc|c} 6 & 1 & 3 \\ 27 & 5 & 9 \\ 26 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right]$$

so $\alpha = 2$ and $\beta = -9$ $\therefore [\vec{b}_1]_{\mathcal{P}} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$ NOT $\begin{bmatrix} 2 \\ -9 \\ 0 \end{bmatrix}$!

1B: Find the change-of-basis matrix M from $\mathcal{B}$ to $\mathcal{P}$.

$$M = \left[[\vec{b}_1]_{\mathcal{P}} \;\; [\vec{b}_2]_{\mathcal{P}} \right]$$

so we need $[\vec{b}_2]_{\mathcal{P}}$: $\left[\begin{array}{cc|c} 6 & 1 & 1 \\ 27 & 5 & 2 \\ 26 & 5 & 1 \end{array}\right] \sim \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$

so $[b_2]_{\mathcal{P}} = \begin{bmatrix} 1 \\ -5 \end{bmatrix}$

$$\therefore M = \begin{bmatrix} 2 & 1 \\ -9 & -5 \end{bmatrix}$$

2. Let $K = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 3 & 3 & 8 & 0 \\ 4 & 6 & 7 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$.

Make good use of the 0's in the matrix K to find the determinant of K using the "cofactor expansion" method we've used in class. Show all the intermediate steps and results. (You can check your final answer on your calculator, of course).

Let's use the 1st row:

$$|K| = 1\begin{vmatrix} 3 & 8 & 0 \\ 6 & 7 & 0 \\ 0 & 5 & 2 \end{vmatrix} - 0\left|\text{it doesn't matter}\right| + 0\left|\text{it doesn't matter}\right| - 4\begin{vmatrix} 3 & 3 & 8 \\ 4 & 6 & 7 \\ 0 & 0 & 5 \end{vmatrix}$$

$$= 1\left(2\begin{vmatrix} 3 & 8 \\ 6 & 7 \end{vmatrix}\right) - 0 + 0 - 4\left(5\begin{vmatrix} 3 & 3 \\ 4 & 6 \end{vmatrix}\right)$$

$$= 1\cdot 2\cdot(21-48) - 4\cdot 5(18-12)$$

$$= 1\cdot 2\cdot(-27) - 4\cdot 5\cdot(+6) = -54 - 120 = -174$$

3. Let $S = \begin{bmatrix} 1 & -3 \\ 4 & 8 \end{bmatrix}$. Find the characteristic polynomial of S, and the eigenvalues of S.

$$\text{char. poly} = \det(S - \lambda I) = \begin{vmatrix} 1-\lambda & -3 \\ 4 & 8-\lambda \end{vmatrix} = (1-\lambda)(8-\lambda) + 12$$

$$= 8 - 9\lambda + \lambda^2 + 12 = \lambda^2 - 9\lambda + 20 = (\lambda-5)(\lambda-4)$$

and the eigenvalues are $\lambda = 5$ or $\lambda = 4$