1. Suppose $A = \begin{bmatrix} -17 & 50 \\ -10 & 28 \end{bmatrix}$

the system $\begin{cases} 5\lambda = 115 \\ 4\lambda = 62 \end{cases}$ is inconsistent!

1A: Is $\mathbf{c} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$ an eigenvector of A? If so, what's the eigenvalue? If not, why not?

Let's "test" it: Does $A\vec{c} = \lambda\vec{c}$ for some λ?

Here $A\vec{c} = \begin{bmatrix} -17 & 50 \\ -10 & 28 \end{bmatrix}\begin{bmatrix} 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 115 \\ 62 \end{bmatrix}$, which is not a scalar multiple of $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$.

1B: Is $\mathbf{c} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ an eigenvector of A? If so, what's the eigenvalue? If not, why not?

Here $A\vec{c} = \begin{bmatrix} -17 & 50 \\ -10 & 28 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so $\begin{bmatrix} 15 \\ 6 \end{bmatrix}$ is a scalar multiple of $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ and that scalar is the eigenvalue $\lambda = 3$.

2. Let $M = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 1 & 5 \\ 2 & 1 & 4 \end{bmatrix}$; then the first two column vectors $\mathbf{m}_1$ and $\mathbf{m}_2$ of M are a basis $\mathcal{B}$ of Col(M).

Another basis for M (you don't have to check this) is $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where $\mathbf{c}_1 = \begin{bmatrix} 9 \\ 11 \\ 8 \end{bmatrix}$ and $\mathbf{c}_2 = \begin{bmatrix} 6 \\ 10 \\ 7 \end{bmatrix}$.

2A: Let $\mathbf{k} = M\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ (this is a LC of the columns of M). What is $\mathbf{k}$ explicitly?

$M\begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$ by calculator is $\vec{K} = \begin{bmatrix} 29 \\ 23 \\ 18 \end{bmatrix}$

2B: Find each of the following: $[\mathbf{k}]_{\mathcal{B}}$ $[\mathbf{k}]_{\mathcal{C}}$

$[\vec{K}]_{\mathcal{B}}$ is the vector of weights required to write $\vec{K}$ as a L.C. of $\vec{m}_1$ and $\vec{m}_2$. rref of $[\vec{m}_1\ \vec{m}_2 | \vec{K}]$ shows $5\vec{m}_1 + 8\vec{m}_2 = \vec{K}$, so $[\vec{K}]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$[\vec{K}]_{\mathcal{C}}$ is the vector of weights req'd to write $\vec{K} = \alpha_1\vec{c}_1 + \alpha_2\vec{c}_2$; rref $[\vec{c}_1\ \vec{c}_2 | \vec{K}]$ shows $[\vec{K}]_{\mathcal{C}} = \begin{bmatrix} 19/3 \\ -14/3 \end{bmatrix}$

2C: Find each of these: $[\mathbf{m}_1]_{\mathcal{B}}$ $[\mathbf{m}_2]_{\mathcal{B}}$

Since $\vec{m}_1 = 1\vec{m}_1 + 0\vec{m}_2$ we have $[\vec{m}_1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; similarly $\vec{m}_2 = 0\vec{m}_1 + 1\vec{m}_2 \Rightarrow [\vec{m}_2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

→ SOLUTION STARTS HERE →

2D: Find the change of basis matrix P from $\mathcal{B}$ to $\mathcal{C}$.

This is $\overset{P}{\left[[\vec{m}_1]_{\mathcal{C}}\ [\vec{m}_2]_{\mathcal{C}} \right]}$; we need to solve two equations to show how to write $\vec{m}_1$ and then $\vec{m}_2$ as L.C.'s of the basis elts. in $\mathcal{C}$. We've discussed that one rref needs to be formed:

$$\text{rref}\left[\begin{array}{cc|c|c} 9 & 6 & 1 & 3 \\ 11 & 10 & 3 & 1 \\ 8 & 7 & 2 & 1 \end{array}\right] = \left[\begin{array}{cc|c|c} 1 & 0 & -1/3 & 1 \\ 0 & 1 & 2/3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} m_1 = -\tfrac{1}{3}\vec{c}_1 + \tfrac{2}{3}\vec{c}_2 \Rightarrow [\vec{m}_1]_{\mathcal{C}} = \begin{bmatrix} -1/3 \\ 2/3 \end{bmatrix} \\ m_2 = \vec{c}_1 - \vec{c}_2 \Rightarrow [\vec{m}_2]_{\mathcal{C}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{array} \Rightarrow \text{COB is matrix } \begin{bmatrix} -1/3 & 1 \\ 2/3 & -1 \end{bmatrix}$$

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2E: Show that your matrix P "changes" $[\mathbf{k}]_{\mathcal{B}}$ into $[\mathbf{k}]_{\mathcal{C}}$ appropriately.

We're asked to verify that

$$P[\vec{K}]_{\mathcal{B}} = [\vec{K}]_{\mathcal{C}}.$$

But,

$$\begin{bmatrix} -1/3 & 1 \\ 2/3 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \end{bmatrix} = \begin{bmatrix} -5/3 + 8 \\ 10/3 - 8 \end{bmatrix} = \begin{bmatrix} \frac{-5+24}{3} \\ \frac{10-24}{3} \end{bmatrix} = \begin{bmatrix} 19/3 \\ -14/3 \end{bmatrix}$$

which is $[\vec{K}]_{\mathcal{C}}$ (see prob. 2B)