Calculus I **EXAM 2**
MATH 105 **March 20, 2015**

Name: KEY

SHOW ALL WORK, CLEARLY AND LEGIBLY, TO RECEIVE FULL CREDIT. CORRECT SPELLING, ORGANIZATION OF YOUR SOLUTION, AND PROPER USE OF MATHEMATICAL NOTATION ALL COUNT. YOU MAY USE A STAND-ALONE GRAPHING CALCULATOR, BUT NOT ANY INTERNET-BASED CALCULATORS. NO NOTES, BOOKS, OR OTHER ADDITIONAL RESOURCES ARE PERMITTED. GOOD LUCK!

1.) (10 pts.) Use the given table to compute and evaluate the derivatives of the functions below.

x	1	4	6
$f(x)$	4	0	6
$f'(x)$	5	7	4
$g(x)$	4	1	6
$g'(x)$	5	1/2	3

a.) Compute the derivative of $y = f(g(x))$ when $x = 6$.

$y' = f'(g(x)) \cdot g'(x)$

At $x = 6$:

$f'(6) \cdot 3 = 4 \cdot 3 = 12$

b.) Compute the derivative of $y = g(f(x))$ when $x = 1$.

$y' = g'(f(x)) \cdot f'(x)$

At $x = 1$:

$g'(4) \cdot 5 = \frac{1}{2} \cdot 5 = \frac{5}{2}$

2.) (15 pts.) Differentiate implicitly and solve for $\frac{dy}{dx}$:

$$x^2 + \sin y = xy^2 + 1.$$

$$2x + \cos y \frac{dy}{dx} = y^2 \cdot 1 + x \cdot 2y \cdot \frac{dy}{dx}$$

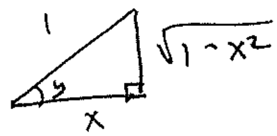
$$\cos y \frac{dy}{dx} - 2xy \frac{dy}{dx} = y^2 - 2x$$

$$\frac{dy}{dx}\left(\cos y - 2xy\right) = y^2 - 2x$$

$$\frac{dy}{dx} = \frac{y^2 - 2x}{\cos y - 2xy}$$

3.) (15 pts.)

a.) (5 pts.) Use a reference triangle to simplify $\tan(\underbrace{\cos^{-1} x}_{=y})$ to a quantity containing no trigonometric or inverse trigonometric functions.



$$\tan y = \frac{\sqrt{1-x^2}}{x}$$

b.) (5 pts.) Compute y' if $y = \sin^{-1}(\sqrt{x})$. Show each part of your derivative. You do not need to simplify.

$$y' = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \cdot \frac{1}{2} x^{-\frac{1}{2}}$$

c.) (5 pts.) Find an antiderivative of $f(x) = \dfrac{3x^2}{1+x^6}$. Be sure to confirm it is a valid antiderivative.

$$F(x) = \tan^{-1}(x^3)$$

Check: $F'(x) = \dfrac{1}{1+(x^3)^2} \cdot 3x^2 = \dfrac{3x^2}{1+x^6} = f(x)$

✓

4.) (15 pts.)

a.) (5 pts.) Compute y' if $y = \dfrac{3^{\cos x}}{\ln x + 4x^2}$. Show each part of your derivative. You do not need to simplify.

$$y' = \frac{(\ln x + 4x^2) \cdot 3^{\cos x} \ln 3 \cdot (-\sin x) - 3^{\cos x} \cdot \left(\frac{1}{x} + 8x\right)}{(\ln x + 4x^2)^2}$$

b.) (5 pts.) Determine whether $G(x) = x \ln x - x$ is an antiderivative of $g(x) = \ln x$. Justify your response.

$$G'(x) = (\ln x)(1) + (x)\left(\frac{1}{x}\right) - 1$$

$$= \ln x + 1 - 1$$

$$= \ln x = g(x)$$

So: yes, $G(x)$ <u>is</u> an antiderivative of $g(x)$.

c.) (5 pts.) Find an antiderivative of $h(x) = \dfrac{3x^2 + \cos x}{x^3 + \sin x}$. Be sure to confirm it is a valid antiderivative.

$$H(x) = \ln(x^3 + \sin x)$$

Check: $H'(x) = \dfrac{1}{x^3 + \sin x} \cdot (3x^2 + \cos x) = \dfrac{3x^2 + \cos x}{x^3 + \sin x} = h(x$

5.) (15 pts.) Compute the following limits. You may use any algebra- or calculus-based method to do so. You may check your answer with a graph, but a graph alone is insufficient justification for a response and will earn no credit.

a.) (5 pts.) $\lim_{x\to 0} \frac{7x^2+4x+1}{9+3x^2}$ $= \frac{0+0+1}{9+0}$

$= \frac{1}{9}$

b.) (5 pts.) $\lim_{x\to\infty} \left(\frac{7x^2+4x+1}{9+3x^2}\right)\left(\frac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$

$= \lim_{x\to\infty} \frac{7+\overset{\to 0}{\frac{4}{x}}+\overset{\to 0}{\frac{1}{x^2}}}{\underset{\to 0}{\frac{9}{x^2}}+3}$

$= \lim_{x\to\infty} \frac{7}{3} = \boxed{\frac{7}{3}}$

c.) (5 pts.) $\lim_{x\to 1} \frac{e^x-e}{\ln x}$ $= \lim_{x\to 1} \frac{e^x}{\frac{1}{x}} = \lim_{x\to 1} x e^x$

$= 1 \cdot e^1$

$= \boxed{e}$

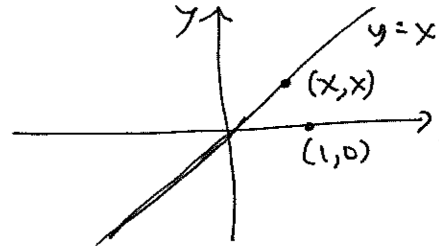
As $x \to 1$:

$e^x - e \to 0$

$\ln x \to 0$

So: L'Hôpital's Rule can be used.

6.) (15 pts.) *Optimization:* Find the point on the line $y = x$ closest to the point $(1, 0)$. Use calculus to solve this and use the First or Second Derivative Test to confirm your result.



Minimize Distance

$$l = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

$$l = \sqrt{(x-1)^2 + (x-0)^2}$$

$$l = \sqrt{x^2 - 2x + 1 + x^2}$$

$$l = \sqrt{2x^2 - 2x + 1}$$

$$l' = \frac{1}{2}(2x^2 - 2x + 1)^{-\frac{1}{2}} \cdot (4x - 2)$$

$$l' = \frac{4x-2}{2\sqrt{2x^2-2x+1}} = 0$$

So: $4x - 2 = 0$

$$\boxed{x = \frac{1}{2}}$$

First Derivative Test:

l	↘		↗
l'	⊖	0	⊕
x	0	$\frac{1}{2}$	1

$$l'(0) = \frac{-2}{2\sqrt{1}} = -1 < 0$$

$$l'(1) = \frac{4-2}{2\sqrt{2-2+1}} = \frac{2}{2\sqrt{1}} = 1 > 0$$

So: local min at $x = \frac{1}{2}$.

The point is $\boxed{\left(\frac{1}{2}, \frac{1}{2}\right)}$.

7.) (15 pts.) Consider the function $y = 4 - 2x^2 + \frac{1}{6}x^4$.

a.) (5 pts.) Use calculus to compute all stationary points of y.

$$y' = -4x + \frac{2}{3}x^3 = 0$$

$$\frac{2}{3}x^3 = 4x$$ $x=0$ satisfies this.

$$\frac{2}{3}x^2 = 4$$

$$x^2 = 6$$ $x = \pm\sqrt{6}$ also are stationary points

b.) (5 pts.) Use the First or Second Derivative Test to determine whether each stationary point is a local maximum, local minimum, or neither.

$$y'' = -4 + 2x^2$$

$y''(-\sqrt{6}) = -4+2(6) = 8 > 0$ so local min at $x = -\sqrt{6}$

$y''(0) = -4 < 0$ so local max at $x = 0$

$y''(\sqrt{6}) = -4+2(6) = 8 > 0$ so local min at $x = \sqrt{6}$

c.) (5 pts.) Use calculus to compute all points of inflection of y.

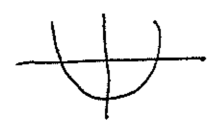
$y'' = 0$: $\quad -4 + 2x^2 = 0$

$$2x^2 = 4$$

$$x = 2$$

$$x = \pm\sqrt{2}$$

$-4+2x^2$:



Changes between $\ominus$ and $\oplus$ at $x = \pm\sqrt{2}$, so these are inflection points.

BONUS (5 pts.): Submit a math joke, math poem, or original creative mathematically-themed drawing. It is permissible to look up other people's math jokes or poems, but if you do so, cite your source. You may use the back of this page.