

Math 105: Review for Exam II - Solutions

1. Find  $dy/dx$  for each of the following.

(a)  $y = x^2 + 2^x + e^2 + e^{2x} + \ln 2 + \ln(2x) + (\ln 2)x + \arctan 2$

$$\frac{dy}{dx} = 2x + (\ln 2)2^x + 2e^{2x} + \frac{1}{2x} \cdot 2 + \ln 2 \quad \text{Note that } e^2, \ln 2, \text{ and } \arctan 2 \text{ are constants.}$$

(b)  $y = \sqrt{x} \cdot \arctan(5x)$

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \arctan(5x) + \sqrt{x} \cdot \frac{1}{1+(5x)^2} \cdot 5 = \frac{\arctan(5x)}{2x^{1/2}} + \frac{5\sqrt{x}}{1+25x^2}$$

(c)  $y = \ln(\tan(2^{\cos(x^2)}))$

$$\frac{dy}{dx} = \frac{1}{\tan(2^{\cos(x^2)})} \cdot \sec^2(2^{\cos(x^2)}) \cdot \ln 2(2^{\cos(x^2)}) \cdot (-\sin(x^2)) \cdot 2x$$

(d)  $y = \frac{x + e^\pi}{\cos 4 + \sin^5(6x)}$

Note that  $e^\pi$  and  $\cos 4$  are constants.

$$\frac{dy}{dx} = \frac{(1)(\cos 4 + \sin^5(6x)) - (x + e^\pi)(5 \sin^4(6x) \cdot \cos(6x) \cdot 6)}{(\cos 4 + \sin^5(6x))^2} \quad \text{Recall that } \sin^5(6x) = (\sin(6x))^5.$$

(e)  $y = (x^2 + 1)^{\sin x}$

We need logarithmic differentiation here.

$$\ln y = \sin x \cdot \ln(x^2 + 1)$$

Take natural log of each side.

$$\frac{1}{y} \frac{dy}{dx} = \cos x \cdot \ln(x^2 + 1) + \sin x \cdot \frac{1}{x^2 + 1} \cdot 2x$$

Differentiate.

$$\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] y$$

Solve for  $\frac{dy}{dx}$ .

$$\frac{dy}{dx} = \left[ \cos x \cdot \ln(x^2 + 1) + \frac{2x \sin x}{x^2 + 1} \right] \cdot (x^2 + 1)^{\sin x}$$

Replace  $y$ .

2. Consider the curve defined by  $x^3 + y^3 = \frac{9}{2}xy$  (known as the Folium of Descartes).

(a) Find  $dy/dx$ .

Use implicit differentiation.

$$\begin{aligned} 3x^2 + 3y^2 \frac{dy}{dx} &= \frac{9}{2}y + \frac{9}{2}x \frac{dy}{dx} \\ 3y^2 \frac{dy}{dx} - \frac{9}{2}x \frac{dy}{dx} &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} \left( 3y^2 - \frac{9}{2}x \right) &= \frac{9}{2}y - 3x^2 \\ \frac{dy}{dx} &= \frac{\frac{9}{2}y - 3x^2}{3y^2 - \frac{9}{2}x} \end{aligned}$$

(b) Verify that the point (1,2) is on the curve above.

We must check to see if the values  $x = 1$  and  $y = 2$  satisfy the equation above.

$$\begin{aligned} x^3 + y^3 &\stackrel{?}{=} \frac{9}{2}xy \\ 1^3 + 2^3 &\stackrel{?}{=} \frac{9}{2} \cdot 1 \cdot 2 \\ 9 &\stackrel{?}{=} 9 \end{aligned}$$

Thus, the point (1,2) is on the curve.

- (c) **Find the equation of the tangent line at the point (1,2).**

We want  $y = mx + b$ .

$$m = \frac{\frac{9}{2} \cdot 2 - 3 \cdot 1^2}{3 \cdot 2^2 - \frac{9}{2} \cdot 1} = \frac{4}{5}, \text{ so } y = \frac{4}{5}x + b.$$

Now plug in  $x = 1$  and  $y = 2$  to find  $b$ .

$$2 = \frac{4}{5} \cdot 1 + b \Rightarrow \frac{6}{5} = b$$

Therefore, we have  $y = \frac{4}{5}x + \frac{6}{5}$ .

3. **Evaluate the following limits.**

Throughout this solution, the symbol  $\star$  will stand for whatever notation your instructor prefers for using L'Hopital's Rule on the indeterminate form  $0/0$ ; this may be  $\stackrel{0/0}{=}$  or  $\stackrel{L'H}{=}$  or  $\stackrel{H}{=}$  or  $\stackrel{0/0}{\sim}$  or "has the form  $\frac{0}{0}$ , and so, by L'Hopital's Rule, is equal to" or something else. The symbol  $\heartsuit$  will serve the same purpose for the indeterminate forms  $\infty/\infty$  and  $-\infty/\infty$ .

(a)  $\lim_{x \rightarrow 1} \frac{x^3 - 1}{7 - 7x} \star \lim_{x \rightarrow 1} \frac{3x^2}{-7} = \frac{3}{-7} = -\frac{3}{7}$

(b)  $\lim_{x \rightarrow 0} \frac{1 - \cos(2x)}{3^x} = \frac{0}{1} = 0$

Can't use (and don't need) L'Hopital's Rule!

(c)  $\lim_{x \rightarrow 0} \frac{1 - \cos(4x)}{5x^2} \star \lim_{x \rightarrow 0} \frac{4 \sin(4x)}{10x} \star \lim_{x \rightarrow 0} \frac{16 \cos(4x)}{10} = \frac{16}{10} = \frac{8}{5}$

(d)  $\lim_{x \rightarrow \infty} \frac{x^2}{2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2x}{\ln 2 \cdot 2^x} \heartsuit \lim_{x \rightarrow \infty} \frac{2}{\ln 2 \cdot \ln 2 \cdot 2^x} = 0$

4. **Consider the function  $f(x) = x^4 e^x$  with domain all real numbers.**

- (a) **Find the  $x$ -value(s) of all roots ( $x$ -intercepts) of  $f$ .**

The equation  $x^4 e^x = 0$  means  $x^4 = 0$  (that is,  $x = 0$ ) or  $e^x = 0$  (no solution), so the only root is at  $x = 0$ .

- (b) **Find the  $x$ - and  $y$ -value(s) of all critical points and identify each as a local max, local min, or neither.**

$$\begin{aligned} f'(x) &= 4x^3 e^x + x^4 e^x \\ 0 &= x^3 e^x (4 + x) \\ \Rightarrow x &= 0, -4 \end{aligned}$$

Note that  $e^x$  is never 0.

	$x < -4$	$-4 < x < 0$	$4 < x$
$f'$	positive	negative	positive
$f$	$\nearrow$	$\searrow$	$\nearrow$

$y$ -values:  $f(-4) = 256e^{-4} \approx 4.689$ ,  $f(0) = 0$

So,  $f$  has a local maximum at  $(-4, 256e^{-4})$  and a local minimum at  $(0, 0)$ .

- (c) **Find the  $x$ - and  $y$ -value(s) of all global extrema and identify each as a global max or global min.**

There is a global minimum at  $(0, 0)$ . There is no global maximum because as  $x \rightarrow \infty$ ,  $f(x) \rightarrow \infty$ . Note that as  $x \rightarrow -\infty$ ,  $f(x) \rightarrow 0$ . You can verify this by using L'Hopital's Rule on  $x^4/e^{-x}$ .

(d) Find the  $x$ -value(s) of all inflection points.

$$f''(x) = 12x^2e^x + 4x^3e^x + 4x^3e^x + x^4e^x \quad \text{Use Product Rule on each product in } f'(x) \text{ above.}$$

$$0 = e^x(x^4 + 8x^3 + 12x^2)$$

$$0 = e^x x^2(x^2 + 8x + 12)$$

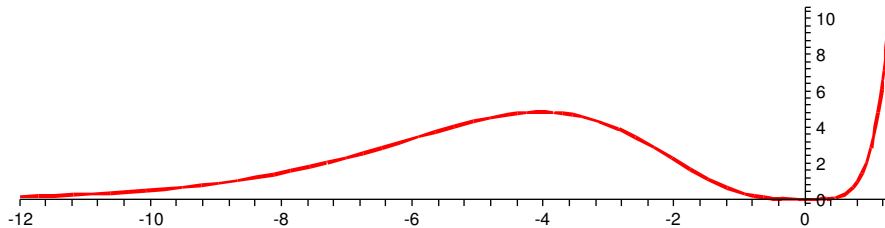
$$0 = e^x x^2(x + 2)(x + 6)$$

$$\Rightarrow x = 0, -2, -6$$

	$x < -6$	$-6 < x < -2$	$-2 < x < 0$	$0 < x$
$f''$	positive	negative	positive	positive
$f$	concave up	concave down	concave up	concave up

So, the  $x$ -values of the inflection points of  $f$  are  $x = -2$  and  $x = -6$  but NOT  $x = 0$ .

(e) Sketch  $f$ .



5. How would your answers to the previous question change if the domain of  $f$  were  $[-10, 10]$ ?

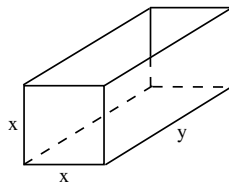
There would be a global maximum at  $(10, 10^4 e^{10})$ . (And the graph would be restricted to  $-10 \leq x \leq 10$ ).

6. [Omit this problem.] Use the Intermediate Value Theorem to explain why  $f(x) = x^3 - 4x^2 + 5$  must have a root somewhere on the interval  $[1, 2]$ .

7. Find an antiderivative of  $y = 5^x + x^3 + \cos(2x) + e^3$ .

The answer is  $\frac{5^x}{\ln 5} + \frac{x^4}{4} + \frac{\sin 2x}{2} + e^3 x + C$ , where  $C$  is any constant.

8. You are planning to build a box-shaped aquarium with no top and with two square ends. Your budget is \$288. If the glass for the sides costs \$12 per square foot and the opaque material for the bottom costs \$3 per square foot, what dimensions will maximize the volume? Be sure to show how you know you have found the maximum.



Goal: Maximize volume

Objective function: volume =  $V = x \cdot x \cdot y = x^2 y$

We need to get this down to a function of just one variable, so we use the *constraint equation*:

$$\begin{aligned}\text{total cost} &= (\text{cost of base}) + (\text{cost of two square ends}) + (\text{cost of two other sides}) \\ 288 &= 3xy + 12 \cdot 2x^2 + 12 \cdot 2xy \\ 288 &= 27xy + 24x^2 \\ 288 - 24x^2 &= 27xy \\ \frac{288 - 24x^2}{27x} &= y\end{aligned}$$

Substituting this back into the objective function gives

$$V = x^2y = x^2 \cdot \frac{288 - 24x^2}{27x} = x \cdot \frac{288 - 24x^2}{27} = \frac{1}{27}(288x - 24x^3).$$

Now that we have  $V$  as a function of just one variable, we find its maximum.

$$\begin{aligned}V'(x) &= \frac{1}{27}(288 - 72x^2) \\ 0 &= \frac{1}{27}(288 - 72x^2) \\ 0 &= (288 - 72x^2) \\ 72x^2 &= 288 \\ x^2 &= \frac{288}{72} \\ x &= 2\end{aligned}$$

We discard  $x = -2$  because lengths must be nonnegative.

Since  $V'$  is positive for  $x < 2$  and negative for  $2 < x$ , we know that the maximum occurs at  $x = 2$ .

And  $y = \frac{288 - 24x^2}{27x} = \frac{288 - 24 \cdot 2^2}{27 \cdot 2} = \frac{32}{9}$ , so the dimensions are 2 by 2 by  $\frac{32}{9}$ .