Calculus I
MATH 105

EXAM 2
March 16, 2011

Name: KEY

YOUR GRADE IS BASED ON CORRECTNESS, COMPLETENESS, AND CLARITY ON EACH EXERCISE. EXPLAIN ALL ANSWERS COMPLETELY. YOU MAY USE A CALCULATOR, BUT NO NOTES, BOOKS, OR OTHER STUDENTS. GOOD LUCK!

1.) (10 pts.) Use the given table to compute and evaluate the derivatives of the functions below.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	7	9	5
9	5	3	1	7

a.) Compute the derivative of $y = xf(x)$, then evaluate when $x = 1$.

$y' = f(x) \cdot 1 + x \cdot f'(x)$

At $x = 1$, $y' = 3 \cdot 1 + 1 \cdot 7$

$= 3 + 7$

$= \boxed{10}$

b.) Compute the derivative of $y = f(g(x))$, then evaluate when $x = 1$.

$y' = f'(g(x)) \cdot g'(x)$

At $x = 1$, $y' = f'(9) \cdot 5$

$= 3 \cdot 5$

$= \boxed{15}$

2.) (15 pts.) Compute the derivative of $y = \dfrac{x^2+4x+3}{x+3}$ two ways. Then compare your answers.

a.) (5 pts.) Do not simplify first. Use the Quotient Rule. Simplify AFTER computing the derivative.

$$\frac{(x+3)(2x+4) - (x^2+4x+3)(1)}{(x+3)^2}$$

$$= \frac{2x^2+10x+12 - x^2-4x-3}{(x+3)^2}$$

$$= \frac{x^2+6x+9}{(x+3)^2} = \boxed{1}$$

b.) (5 pts.) Simplify the fraction FIRST. Then use the Power Rule to compute the derivative.

$$y = \frac{(x+3)(x+1)}{x+3} = x+1$$

$$y' = \boxed{1}$$

c.) (5 pts.) Compare your two fully simplified final answers. Are they the same? *Should* they be?

They are the same. They should be.

However you compute the derivative, your answers are <u>equivalent</u>.

3.) (15 pts.)

a.) (5 pts.) Use implicit differentiation to compute the derivative of $x^2y + y^3 = 75$.

$$y \cdot 2x + x^2 \cdot \frac{dy}{dx} + 3y^2 \cdot \frac{dy}{dx} = 0$$

b.) (5 pts.) Solve for $\frac{dy}{dx}$.

$$(x^2 + 3y^2)\frac{dy}{dx} = -2xy$$

$$\frac{dy}{dx} = \boxed{\frac{-2xy}{x^2+3y^2}}$$

c.) (5 pts.) Determine the slope of the tangent line to the curve in part (a) at the point $(x, y) = (4, 3)$.

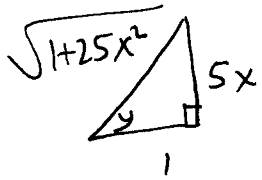
$$\frac{dy}{dx} \text{ at } (4,3) = \frac{-2 \cdot 4 \cdot 3}{4^2 + 3(3^2)}$$

$$= \boxed{\frac{-24}{43}}$$

4.) (15 pts.)

a.) (5 pts.) Use a reference triangle to write $\cos(\arctan(5x))$ as an algebraic expression, that is, one not involving any trigonometric functions. (Note: you are not being asked to compute a derivative here.)

Let $\arctan(5x) = y$:



Then $\cos y = \boxed{\dfrac{1}{\sqrt{1+25x^2}}}$

b.) (5 pts.) Compute the derivative of $\ln(\arcsin(x^2))$.

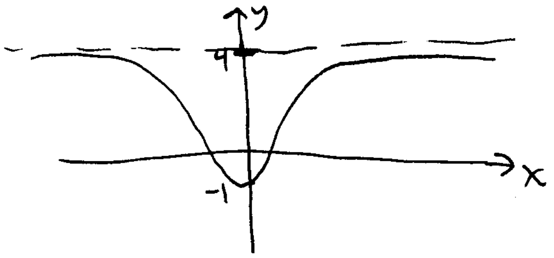
$$\frac{1}{\arcsin(x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

c.) (5 pts.) Compute an antiderivative of $g(x) = \dfrac{e^x + \sec^2 x + 2x}{e^x + \tan x + x^2 + \pi}$.

$$G(x) = \ln\left(e^x + \tan x + x^2 + \pi\right)$$

5.) (15 pts.)

a.) (5 pts.) Sketch a graph (it is fine to use your calculator) of $y = \frac{40x^4+4x^2-1}{10x^4+8x^2+1}$ and use your graph to make a guess about $\lim\limits_{x\to\infty} \dfrac{40x^4+4x^2-1}{10x^4+8x^2+1}$. (Be sure to show *how* you are using your graph to make your guess.)



As $x \to \infty$, y-values approach the asymptote $y = 4$.

So the limit appears to be $\boxed{4}$

b.) (5 pts.) Use an algebraic or calculus-based technique to confirm your guess from part (a).

$$\lim_{x\to\infty} \frac{40x^4+4x^2-1}{10x^4+8x^2+1}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^4}\left(40 + \overset{\to 0}{\frac{4}{x^2}} - \overset{\to 0}{\frac{1}{x^4}}\right)}{\cancel{x^4}\left(10 + \underset{\to 0}{\frac{8}{x^2}} + \underset{\to 0}{\frac{1}{x^4}}\right)} = \frac{40}{10} = \boxed{4}$$

c.) (5 pts.) Use L'Hôpital's Rule to compute $\lim\limits_{x\to 0} \dfrac{e^x - 1}{x^2+3x}$. Be sure to confirm that this is a case in which you *can* use L'Hôpital's Rule.

As $x \to 0$:

$e^x - 1 \to e^0 - 1 = 0$

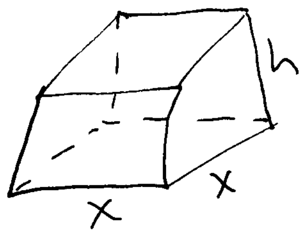
$x^2 + 3x \to 0^2 + 3\cdot 0 = 0$

So: we can use L'Hôpital's Rule

$$= \lim_{x\to 0} \frac{e^x}{2x+3} = \frac{e^0}{2\cdot 0+3}$$

$$= \boxed{\frac{1}{3}}$$

6.) (15 pts.) Suppose an airline policy states that all baggage must be box-shaped with a sum of length, width, and height not exceeding 64 inches. What are the dimensions and volume of a square-based box with the greatest volume under these conditions? (Note: use calculus to solve this and to confirm that your result gives *maximum* volume.)



Maximize volume

Objective: $V = x^2 h$

Constraint: $x + x + h = 64$

$\rightarrow h = 64 - 2x$

Therefore $V = x^2(64-2x) = 64x^2 - 2x^3$

$V' = 128x - 6x^2 = 0$

$2x(64 - 3x) = 0$

$x = 0$ or $\frac{64}{3}$ → Try this one.

↓

V would be 0

2nd Deriv. Test: $V'' = 128 - 12x$

At $x = \frac{64}{3}$, $V'' = 128 - 256 < 0$

So volume is max when

$\boxed{x = \frac{64}{3} \text{ inches}}$

Then $\boxed{h =}\ 64 - 2\left(\frac{64}{3}\right) = \boxed{\frac{64}{3} \text{ inches}}$

and $\boxed{V = \left(\frac{64}{3}\right)^3 \text{ in}^3 \approx 9709 \text{ in}^3}$

7.) (15 pts.) Consider the function $y = \frac{1}{2}x + \sin x$ on the interval $[0, 2\pi]$.

a.) (5 pts.) At which x-values, from 0 to 2π, does y have stationary points? Use calculus to produce your answer, and give the x-values *exactly*, not as decimal approximations.

$y' = \frac{1}{2} + \cos x = 0$

$\cos x = -\frac{1}{2}$

$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

b.) (5 pts.) Demonstrate, using the First Derivative Test and/or the Second Derivative Test, whether each of your stationary points from part (a) is a local maximum or a local minimum.

$y'' = -\sin x$

At $\frac{2\pi}{3}$, $y'' = -\sin\left(\frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} < 0 \rightarrow$ max at $x = \frac{2\pi}{3}$

At $\frac{4\pi}{3}$, $y'' = -\sin\left(\frac{4\pi}{3}\right) = -\left(-\frac{\sqrt{3}}{2}\right) > 0 \rightarrow$ min at $x = \frac{4\pi}{3}$

c.) (5 pts.) How many points of inflection does y have for x-values in the interval $[0, 2\pi]$? Use calculus to support your answer, and as in part (a), give all x-values *exactly*.

$y'' = -\sin x = 0$ at $0, \pi, 2\pi$.

In the interval, y can only change concavity at $x = \pi$.

Let's see if it does: $\left.\begin{array}{l} -\sin\left(\frac{\pi}{2}\right) = -1 \\ -\sin\left(\frac{3\pi}{2}\right) = 1 \end{array}\right\}$ So concavity does change.

BONUS: The mathematical definition of *continuity at a point "a"* involves the concept of a limit. Explain this connection. (You can state the definition of continuity at a point, or you can give a more intuitively-worded explanation that involves limits.) You may use the back of this page if you would like more space to write.